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On powers of countably pracompact groups

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A B S T R A C T

In 1990, Comfort asked: is there, for every cardinal number $\alpha \leq 2^{\mathfrak{c}}$, a topological group G such that G^γ is countably compact for all cardinals $\gamma < \alpha$, but G^α is not countably compact? A similar question can also be asked for countably pracompact groups: for which cardinals α is there a topological group G such that G^γ is countably pracompact for all cardinals $\gamma < \alpha$, but G^α is not countably pracompact? In this paper we construct such group in the case $\alpha = \omega$, assuming the existence of $\mathfrak{c}$ incomparable selective ultrafilters, and in the case $\alpha = \kappa^+$, with $\omega \leq \kappa \leq 2^{\mathfrak{c}}$, assuming the existence of $2^{\mathfrak{c}}$ incomparable selective ultrafilters. In particular, under the second assumption, there exists a topological group G so that $G^{2^{\mathfrak{c}}}$ is countably pracompact, but $G^{(2^{\mathfrak{c}})^+}$ is not countably pracompact, unlike the countably compact case.

1. Introduction

Throughout this paper, every topological space will be Tychonoff (Hausdorff and completely regular) and every topological group will be Hausdorff (thus, also Tychonoff). For an infinite set X, $[X]^{<\omega}$ will denote the family of all finite subsets of X, and $[X]^\omega$ will denote the family of all countable subsets of X. Recall that an infinite topological space X is said to be

- *pseudocompact* if each continuous real-valued function on X is bounded;
- *countably compact* if every infinite subset of X has an accumulation point in X;
- *countably pracompact* if there exists a dense subset D in X such that every infinite subset of D has an accumulation point in X.

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We denote the set of non-principal (free) ultrafilters on ω by ω^*. The following notion was introduced by Bernstein [3]:

Definition 1.1 *([3]).* Let $p \in \omega^*$ and $\{x_n : n \in \omega\}$ be a sequence in a topological space X. We say that $x \in X$ is a *p-limit point* of $\{x_n : n \in \omega\}$ if $\{n \in \omega : x_n \in U\} \in p$ for every neighborhood U of x.

Notice that if X is a Hausdorff space, for each $p \in \omega^*$ a sequence $\{x_n : n \in \omega\} \subset X$ has at most one p-limit point x and we write $x = p - \lim_{n\in\omega} x_n$ in this case.

One may write the compact-like definitions above using the notion of p-limits. In fact, it is not hard to show that $x \in X$ is an accumulation point of a sequence $\{x_n : n \in \omega\} \subset X$ if and only if there exists $p \in \omega^*$ such that $x = p - \lim_{n\in\omega} x_n$. Thus, we have that

- X is countably compact if and only if every sequence $\{x_n : n \in \omega\} \subset X$ has a p-limit, for some $p \in \omega^*$.
- X is countably pracompact if and only if there exists a dense subset D in X such that every sequence $\{x_n : n \in \omega\} \subset D$ has a p-limit in X, for some $p \in \omega^*$.

For pseudocompact spaces, a similar equivalence holds: X is pseudocompact if and only if for every countable family $\{U_n : n \in \omega\}$ of nonempty open sets of X, there exists $x \in X$ and $p \in \omega^*$ such that, for each neighborhood V of x, $\{n \in \omega : V \cap U_n \neq \emptyset\} \in p$.

There are many concepts related to compactness and pseudocompactness which have emerged in the last years. In this paper, we highlight the following, which was introduced in [8].

Definition 1.2 *([8]).* A topological space X is called *selectively pseudocompact*[3] if for each sequence $\{U_n : n \in \omega\}$ of nonempty open subsets of X there is a sequence $\{x_n : n \in \omega\} \subset X$, $x \in X$ and $p \in \omega^*$ such that $x = p - \lim_{n\in\omega} x_n$ and, for each $n \in \omega$, $x_n \in U_n$.

It is clear that every selectively pseudocompact space is pseudocompact and every countably pracompact space is selectively pseudocompact. Also, it was proved in [9] that there exists a pseudocompact topological group which is not selectively pseudocompact, and in [17] that there exists a selectively pseudocompact group which is not countably pracompact.

We shall now briefly recall the definitions and some facts about selective ultrafilters and the Rudin-Keisler order.

Definition 1.3. A *selective ultrafilter* on ω is a free ultrafilter p on ω such that for every partition $\{A_n : n \in \omega\}$ of ω, either there exists $n \in \omega$ such that $A_n \in p$ or there exists $B \in p$ such that $|B \cap A_n| = 1$ for every $n \in \omega$.

Given an ultrafilter p on ω and a function $f : \omega \to \omega$, note that

$$f_*(p) \doteq \{A \subset \omega : f^{-1}(A) \in p\}$$

is also an ultrafilter on ω. Consider then the following definition.

Definition 1.4. Given $p, q \in \omega^*$, we say that $p \leq_{RK} q$ if there exists a function $f : \omega \to \omega$ so that $f_*(q) = p$. Such relation on ω^* is a preorder called the *Rudin-Keisler order*.

[3] This concept was originally defined under the name *strong pseudocompactness*, but later the name was changed, since there were already two different properties named in the previous way (in [1] and [7]).

We say that $p, q \in \omega^*$ are:

- *incomparable* if neither $p \leq_{RK} q$ or $q \leq_{RK} p$;
- *equivalent* if $p \leq_{RK} q$ and $q \leq_{RK} p$.

The existence of selective ultrafilters is independent of ZFC. In fact, there exists a model of ZFC in which there are no P-points[4] in ω^* [19], while Martin's axiom (MA) implies the existence of $2^{\mathfrak{c}}$ incomparable selective ultrafilters [4].

Pseudocompactness is not preserved under products for arbitrary topological spaces [15], but interestingly Comfort and Ross proved that the product of any family of pseudocompact topological groups is pseudocompact [6]. This result motivated Comfort to question whether the product of countably compact groups is also countably compact. More generally, he asked the following question [5]:

Question 1.5 *([5], Question 477).* Is there, for every (not necessarily infinite) cardinal number $\alpha \leq 2^{\mathfrak{c}}$, a topological group G such that G^γ is countably compact for all cardinals $\gamma < \alpha$, but G^α is not countably compact?

The restriction $\alpha \leq 2^{\mathfrak{c}}$ in the question above is due to the following result:

Theorem 1.6 *([12], Theorem 2.6). Let X be a Hausdorff topological space. The following statements are equivalent:*

(i) every power of X is countably compact;
(ii) $X^{2^{\mathfrak{c}}}$ is countably compact;
(iii) $X^{|X|^\omega}$ is countably compact;
(iv) there exists $p \in \omega^*$ such that X is p-compact.[5]

Van Douwen was the first to prove consistently (under MA) that there are two countably compact groups whose product is not countably compact [18]. Also, Question 1.5 was answered positively in [16], assuming the existence of $2^{\mathfrak{c}}$ selective ultrafilters and that $2^{\mathfrak{c}} = 2^{<2^{\mathfrak{c}}}$. Finally, in 2021, it was proved in ZFC that there are two countably compact groups whose product is not countably compact [13].

It is natural also to ask productivity questions for countably pracompact and selectively pseudocompact groups. In this regard, Garcia-Ferreira and Tomita proved that if p and q are non-equivalent (according to the Rudin-Keisler order in ω^*) selective ultrafilters on ω, then there are a p-compact group and a q-compact group whose product is not selectively pseudocompact [11]. Also, Bardyla, Ravsky and Zdomskyy constructed, under MA, a Boolean countably compact topological group whose square is not countably pracompact [2]. However, the following questions remain unsolved in ZFC.

Question 1.7 *(ZFC).* Is it true that selective pseudocompactness is non-productive in the class of topological groups?

Question 1.8 *(ZFC).* Is it true that countable pracompactness is non-productive in the class of topological groups?

[4] A free ultrafilter $p \in \omega^*$ is a *P-point* if, for every sequence $(A_n)_{n\in\omega}$ of elements of p, there exists $A \in p$ so that $A \setminus A_n$ is finite for each $n \in \omega$. Every selective ultrafilter is a P-point.

[5] Given $p \in \omega^*$, a topological space X is *p-compact* if every sequence of points in X has a p-limit. The product of p-compact spaces is p-compact, for every $p \in \omega^*$.

More generally, one can ask Comfort-like questions, such as Question 1.5, for selectively pseudocompact and countably pracompact groups. In the case of selectively pseudocompact groups, the question is restricted to cardinals $\alpha \leq \omega$, due to the next result.

Lemma 1.9. *If G is a topological group such that G^ω is selectively pseudocompact, then G^κ is selectively pseudocompact for every cardinal $\kappa \geq \omega$.*

Proof. Indeed, let $\kappa \geq \omega$ and $(U_n)_{n\in\omega}$ be a family of open subsets of G^κ. For every $n \in \omega$, there are open subsets $U_n^j \subset G$, for each $j < \kappa$, so that $\prod_{j\in\kappa} U_n^j \subset U_n$ and $U_n^j \neq G$ if and only if $j \in F_n$, for a finite subset $F_n \subset \kappa$. Let $F \doteq \bigcup_{n\in\omega} F_n$. For each $n \in \omega$, consider the open subsets $V_n \doteq \prod_{j\in F_n} U_n^j \times \prod_{j\in F\setminus F_n} G \subset G^F$. By assumption, G^F is selectively pseudocompact, thus there is a sequence $\{y_n : n \in \omega\} \subset G^F$ so that $y_n \in V_n$, for every $n \in \omega$, which has an accumulation point y in G^F. Then, given $g \in G$ arbitrarily, the sequence $\{x_n : n \in \omega\} \subset G^\kappa$ defined coordinatewise, for each $n \in \omega$, by

$$x_n^j \doteq \begin{cases} y_n^j, & \text{if } j \in F \\ g, & \text{if } j \in \kappa \setminus F \end{cases}$$

is such that $x_n \in U_n$ for every $n \in \omega$, and has $x \in G^\kappa$ given by

$$x^j \doteq \begin{cases} y^j, & \text{if } j \in F \\ g, & \text{if } j \in \kappa \setminus F \end{cases}$$

as accumulation point. □

Question 1.10. For which cardinals $\alpha \leq \omega$ is there a topological group G such that G^γ is selectively pseudocompact for all cardinals $\gamma < \alpha$, but G^α is not selectively pseudocompact?

In the case of countably pracompact groups, it is still not known whether there exists a cardinal κ satisfying that: if a topological group G is such that G^κ countably pracompact, then G^α is countably pracompact, for each $\alpha > \kappa$. Thus, there is no restriction to the cardinals α yet:

Question 1.11. For which cardinals α is there a topological group G such that G^γ is countably pracompact for all cardinals $\gamma < \alpha$, but G^α is not countably pracompact?

It is worth observing that if G^ω is **countably compact** and $\kappa \geq \omega$, then

$$\Sigma \doteq \{g \in G^\kappa : |\{\alpha \in \kappa : g^\alpha \neq 0\}| \leq \omega\}$$

is a dense subset of G^κ for which every infinite subset has an accumulation point. Thus, in this case G^κ is countably pracompact.

In [10], under the assumption of CH, the authors showed that for every positive integer $k > 0$, there exists a topological group G for which G^k is countably compact but G^{k+1} is not selectively pseudocompact. Thus, Question 1.10 and Question 1.11 are already solved for finite cardinals under CH. The cardinal $\alpha = \omega$ is the only one for which there are still no consistent answers to the Question 1.10.

In this paper:

- assuming the existence of $\mathfrak{c}$ incomparable selective ultrafilters, we answer Question 1.11 for $\alpha = \omega$;
- assuming the existence of $2^\mathfrak{c}$ incomparable selective ultrafilters, we answer Question 1.11 for each successor cardinal $\alpha = \kappa^+$, with $\omega \leq \kappa \leq 2^\mathfrak{c}$.

We will be dealing with Boolean groups, which are also vector spaces over the field $2 = \{0, 1\}$, and thus we can talk about general linear algebra concepts concerning these groups, such as *linearly independent subsets*. More specifically, if $D \subset 2^{\mathfrak{c}}$ is an infinite set, we will consider $[D]^{<\omega}$ as a Boolean group, with the symmetric difference $\triangle$ as the group operation and $\emptyset$ as the neutral element.

Given $p \in \omega^*$, one may define an equivalence relation on $([D]^{<\omega})^\omega$ by letting $f \equiv_p g$ iff $\{n \in \omega : f(n) = g(n)\} \in p$. We let $[f]_p$ be the equivalence class determined by f and $([D]^{<\omega})^\omega/p$ be $([D]^{<\omega})^\omega/\equiv_p$. Notice that this set has a natural vector space structure (over the field 2). For each $D_0 \in [D]^{<\omega}$, the constant function in $([D]^{<\omega})^\omega$ which takes only the value D_0 will be denoted by $\vec{D_0}$.

2. Auxiliary results

In this section we present the auxiliary results that we will use in the constructions. We start with a simple linear algebra lemma, stated and proved in [17].

Lemma 2.1 *([17]). Let A, B and C be subsets in a Boolean group. Suppose that A is a finite set and that $A \cup C$, $B \cup C$ are linearly independent. Then there exists $B' \subset B$ such that $|B'| \leq |A|$ and $A \cup C \cup (B \setminus B')$ is linearly independent.*

The next two technical lemmas will also be useful.

Lemma 2.2. *Let X be an infinite set and $\{X_0, ..., X_n\}$ be a partition of X. Let also $(x_k)_{k\in\omega}$ and $(y_k)_{k\in\omega}$ be sequences in the Boolean group $[X]^{<\omega}$ so that:*

- *$\{x_k : k \in \omega\} \cup \{y_k : k \in \omega\}$ is linearly independent;*
- *for every $p \in \{0, ..., n\}$, both $\{x_k \cap X_p : k \in \omega\}$ and $\{y_k \cap X_p : k \in \omega\}$ are linearly independent.*

Then, there exist a subsequence $\{k_m : m \in \omega\}$ and $n_0 \in \{0, ..., n\}$ so that

$$\{x_{k_m} \cap X_{n_0} : m \in \omega\} \cup \{y_{k_m} \cap X_{n_0} : m \in \omega\}$$

is linearly independent.

Proof. We shall construct inductively a sequence $(A_0^i)_{i\in\omega}$ of subsets of ω as follows. Firstly, if does not exist $k \in \omega$ so that $\{x_k \cap X_0\} \cup \{y_k \cap X_0\}$ is linearly independent, we put $A_0^0 = \emptyset$. Otherwise, we choose the minimum $k_0 \in \omega$ with this property and put $A_0^0 \doteq \{k_0\}$. Suppose that for $l \in \omega$ we have constructed $A_0^0, ..., A_0^l \subset \omega$ such that:

i) $|A_0^i| \leq i + 1$, for each $i = 0, ..., l$;
ii) $A_0^i \subset A_0^j$ if $0 \leq i \leq j \leq l$;
iii) $\{x_k \cap X_0 : k \in A_0^l\} \cup \{y_k \cap X_0 : k \in A_0^l\}$ is linearly independent.
iv) for each $0 \leq i < l$, $A_0^{i+1} \setminus A_0^i = \emptyset$ if, and only if,

$$\{x_k \cap X_0 : k \in A_0^i\} \cup \{y_k \cap X_0 : k \in A_0^i\} \cup \{x_{\tilde{k}} \cap X_0\} \cup \{y_{\tilde{k}} \cap X_0\}$$

is linearly dependent for every $\tilde{k} > \max(A_0^i)$.

In what follows, we will construct A_0^{l+1}. If does not exist $\tilde{k} \in \omega$, $\tilde{k} > \max(A_0^l)$, so that

$$\{x_k \cap X_0 : k \in A_0^l\} \cup \{y_k \cap X_0 : k \in A_0^l\} \cup \{x_{\tilde{k}} \cap X_0\} \cup \{y_{\tilde{k}} \cap X_0\}$$

is linearly independent, we put $A_0^{l+1} = A_0^l$. Otherwise, we choose the minimum $k_{l+1} \in \omega$ with this property, and put $A_0^{l+1} = A_0^l \cup \{k_{l+1}\}$. In any case, $A_0^0, ..., A_0^{l+1}$ satisfy items i)–iv), and then, by induction, there exists a sequence $(A_0^i)_{i\in\omega}$ satisfying them. Now, let $A_0 \doteq \bigcup_{i\in\omega} A_0^i$. If A_0 is infinite, then $\{x_k \cap X_0 : k \in A_0\} \cup \{x_k \cap X_0 : k \in A_0\}$ is linearly independent, and we are done.

On the other hand, suppose that A_0 is finite. We may repeat the process above for $X_1, ..., X_n$, constructing analogous subsets $A_1, ..., A_n \subset \omega$. If either of them is infinite, we are done.

Suppose then that $A_0, ..., A_n$ are finite sets. By construction, for each $\tilde{k} > \max(A_0 \cup ... \cup A_n)$ and $j = 0, ..., n$,

$$\{x_k \cap X_j : k \in A_j\} \cup \{y_k \cap X_j : k \in A_j\} \cup \{x_{\tilde{k}} \cap X_j\} \cup \{y_{\tilde{k}} \cap X_j\}$$

is linearly dependent. Also, since, for every $j = 0, ..., n$,

$$\mathcal{C}_j \doteq \operatorname{span}(\{x_k \cap X_j : k \in A_j\} \cup \{y_k \cap X_j : k \in A_j\})$$

is finite and both $\{x_k \cap X_j : k \in \omega\}$ and $\{y_k \cap X_j : k \in \omega\}$ are linearly independent, we can fix:

- an infinite subset $A \subset \omega$;
- $c_j \in \mathcal{C}_j$, for each $j = 0, ..., n$,

so that

$$x_{\tilde{k}} \cap X_j = (y_{\tilde{k}} \cap X_j) \triangle c_j,$$

for every $\tilde{k} \in A$ and $j = 0, ..., n$. Thus,

$$x_{\tilde{k}} = (x_{\tilde{k}} \cap X_0)\triangle ... \triangle(x_{\tilde{k}} \cap X_n) = (y_{\tilde{k}} \cap X_0)\triangle ... \triangle(y_{\tilde{k}} \cap X_n)\triangle(c_0\triangle ... \triangle c_n) = y_{\tilde{k}}\triangle(c_0\triangle ... \triangle c_n),$$

for every $\tilde{k} \in A$, which is a contradiction, as $\{x_k : k \in \omega\} \cup \{y_k : k \in \omega\}$ is linearly independent. Hence, $A_0, ..., A_n$ cannot all be finite. □

Lemma 2.3. *Let X be an infinite set, $k > 0$ and $\{(x_n^0, ..., x_n^{k-1}) : n \in \omega\} \subset ([X]^{<\omega})^k$ be a sequence. Then, there are:*

- *elements $d_0, ..., d_{k-1} \in [X]^{<\omega}$;*
- *a subsequence $\{(x_{n_l}^0, ..., x_{n_l}^{k-1}) : l \in \omega\}$;*
- *for some*[6] *$0 \le t \le k$, a sequence $\{(y_{n_l}^0, ..., y_{n_l}^{t-1}) : l \in \omega\} \subset ([X]^{<\omega})^t$;*
- *for each $0 \le s < k$, a function $P_s : t \to 2$,*

satisfying that

i) $x_{n_l}^s = \Big(\sum_{i=0}^{t-1} P_s(i) y_{n_l}^i\Big)\triangle d_s$, *for every $l \in \omega$ and $0 \le s < k$;*
ii) $\{y_{n_l}^i : l \in \omega, 0 \le i < t\}$ *is linearly independent.*

Proof. Fix $q \in \omega^*$, and let

$$\mathcal{M} \doteq \Big\{c \in [X]^{<\omega} : [\vec{c}]_q \in \operatorname{span}(\{[x^0]_q, ..., [x^{k-1}]_q\})\Big\}.$$

[6] If $t = 0$, we understand that there is no such sequence and item i) becomes: $x_{n_l}^s = d_s$, for every $l \in \omega$ and $0 \le s < k$.

It is clear that $\mathcal{M}$ is a finite set, thus let $j \geq 0$ and $\{c^0, ..., c^{j-1}\} \subset \mathcal{M}$ be so that $\{c^0, ..., c^{j-1}\}$ is a basis for $\operatorname{span}(\mathcal{M}) \subset [X]^{<\omega}$. Then, let also $t \geq 0$ and $y^0, ..., y^{t-1} \in ([X]^{<\omega})^\omega$ be so that $\mathcal{B} \doteq \{[\vec{c^0}]_q, ..., [\vec{c^{j-1}}]_q, [y^0]_q, ..., [y^{t-1}]_q\}$ is a basis for $\operatorname{span}(\{[x^0]_q, ..., [x^{k-1}]_q\})$. Hence, there are $A \in q$, $P_s : t \to 2$ and $C_s : j \to 2$, for each $0 \leq s < k$, so that

$$x_n^s = \sum_{i=0}^{t-1} P_s(i) y_n^i \triangle \sum_{i=0}^{j-1} C_s(i) c^i,$$

for every $n \in A$ and $0 \leq s < k$. For each $0 \leq s < k$, let $d_s \doteq \sum_{i=0}^{j-1} C_s(i) c^i$.

We shall prove that there exists an infinite subset $I \subset A$ so that $\{y_n^i : n \in I, 0 \leq i < t\}$ is linearly independent. First, note that for each $c \in [X]^{<\omega}$ and nontrivial function $P : t \to 2$ we have that

$$\sum_{i=0}^{t-1} P(i)[y^i]_q \neq [\vec{c}]_q.$$

Therefore, there exists a subset $A_{P,c} \subset A$, $A_{P,c} \in q$, so that

$$\sum_{i=0}^{t-1} P(i) y_n^i \neq c$$

for each $n \in A_{P,c}$. In particular, we conclude that $\{y_n^i : 0 \leq i < t-1\}$ is linearly independent for every $n \in \bigcap_{\substack{P:t\to 2 \\ P\neq 0}} A_{P,\emptyset} \doteq A_0$. We may choose $n_0 \in A_0$.

Now, suppose that, given $p \geq 1$, for each $l = 0, ..., p-1$ we have constructed $A_l \in q$ and $n_l \in A_l$ so that $\{y_{n_l}^i : 0 \leq l < p, 0 \leq i < t\}$ is linearly independent, $(n_l)_{0 \leq l < p}$ is strictly increasing and $A_l \subset A$. Let $\mathcal{C}_p \doteq \operatorname{span}(\{y_{n_l}^i : 0 \leq l < p, 0 \leq i < t\})$,

$$A_p \doteq \bigcap_{\substack{c \in \mathcal{C}_p \\ P:t\to 2 \\ P\neq 0}} A_{P,c} \ (\subset A),$$

and fix $n_p \in A_p$, $n_p > n_{p-1}$. It is clear that $A_p \in q$ and also $\{y_{n_l}^i : 0 \leq l \leq p, 0 \leq i < t\}$ is linearly independent, by construction. Then, by induction, there are a sequence $(A_l)_{l\in\omega}$ of elements of q and a strictly increasing sequence $(n_l)_{l\in\omega}$ of naturals so that $n_l \in A_l$ and $\{y_{n_l}^i : l \in \omega, 0 \leq i < t\}$ is linearly independent. Furthermore,

$$x_{n_l}^s = \Big(\sum_{i=0}^{t-1} P_s(i) y_{n_l}^i\Big) \triangle d_s,$$

for every $l \in \omega$ and $0 \leq s < k$. □

Next, we enunciate **Lemma 3.5** and **Lemma 3.6** of [16], and an immediate consequence of **Lemma 2.1** of [9].

Lemma 2.4 *([16], Lemma 3.5). Let* p_0 *and* p_1 *be incomparable selective ultrafilters. Let* $\{a_k^j : k \in \omega\} \in p_j$ *be a strictly increasing sequence such that* $a_k^j > k$ *for every* $k \in \omega$ *and* $j \in 2$*. Then there exist subsets* I_0 *and* I_1 *of* ω *such that:*

(i) $\{a_k^j : k \in I_j\} \in p_j$ *for each* $j \in 2$*;*

(ii) $\{[k, a^j_k] : j \in 2, k \in I_j\}$ *are pairwise disjoint intervals of* ω*.*

As a corollary of the previous lemma, we obtain:

Lemma 2.5. *Let* $n > 0$ *and* $\{p_j : j \leq n\}$ *be incomparable selective ultrafilters. Let* $\{a^j_k : k \in \omega\} \in p_j$ *be a strictly increasing sequence such that* $a^j_k > k$ *for every* $k \in \omega$ *and* $j \leq n$*. Then there exists a family* $\{I_j : j \leq n\}$ *of subsets of* ω *such that:*

(i) $\{a^j_k : k \in I_j\} \in p_j$ *for each* $j \leq n$*;*
(ii) $\{[k, a^j_k] : j \leq n, k \in I_j\}$ *are pairwise disjoint intervals of* ω*.*

Proof. We will show that the lemma is true for each $n > 0$ by induction. The case $n = 1$ is just Lemma 2.4.

Suppose that the result is true for a given $n_0 > 0$. We claim that it is also true for $n_0 + 1$. Indeed, let $\{p_j : j \leq n_0 + 1\}$ be incomparable selective ultrafilters and $\{a^j_k : k \in \omega\} \in p_j$ be a strictly increasing sequence such that $a^j_k > k$ for every $k \in \omega$ and $j \leq n_0 + 1$. By hypothesis, there exists a family $\{\tilde{I}_j : j \leq n_0\}$ of subsets of ω so that:

- $\{a^j_k : k \in \tilde{I}_j\} \in p_j$ for each $j \leq n_0$;
- $\{[k, a^j_k] : j \leq n_0, k \in \tilde{I}_j\}$ are pairwise disjoint intervals of ω.

Also, by Lemma 2.4, for each $j \leq n_0$ there exist $I_j \subset \tilde{I}_j$ and $K_j \subset \omega$ so that:

- $\{a^j_k : k \in I_j\} \in p_j$ and $\{a^{n_0+1}_k : k \in K_j\} \in p_{n_0+1}$;
- $\{[k, a^j_k] : k \in I_j\} \cup \{[k, a^{n_0+1}_k] : k \in K_j\}$ are pairwise disjoint intervals of ω.

Then, defining $I_{n_0+1} \doteq \bigcap_{j=0}^{n_0} K_j$, we have that $\{I_j : j \leq n_0 + 1\}$ satisfies the hypothesis we want. Therefore, the lemma is true for every $n > 0$. □

The countable version of the previous result is **Lemma 3.6** of [16]:

Lemma 2.6 *([16], Lemma 3.6). Let* $\{p_j : j \in \omega\}$ *be incomparable selective ultrafilters. Let* $\{a^j_k : k \in \omega\} \in p_j$ *be a strictly increasing sequence such that* $a^j_k > k$ *for each* $k, j \in \omega$*. Then there exists a family* $\{I_j : j \in \omega\}$ *of subsets of* ω *such that:*

(i) $\{a^j_k : k \in I_j\} \in p_j$ *for each* $j \in \omega$*;*
(ii) $\{[k, a^j_k] : j \in \omega, k \in I_j\}$ *are pairwise disjoint intervals of* ω*.*

Lemma 2.7 *([9]). Let* G *be a non-discrete Boolean topological group. Then there exist nonempty open sets* $\{U^j_k : k \in \omega, j \in \omega\}$ *such that if* $u^j_k \in U^j_k$ *for each* $k, j \in \omega$*, then* $\{u^j_k : k, j \in \omega\}$ *is linearly independent.*

The following results ensure the existence of certain homomorphisms, necessary to construct the topological groups we want. Their proofs are based on **Lemma 3.7** and **Lemma 4.1** of [16], and also **Lemma 4.1** of [11].

Lemma 2.8. *Let:*

- E *be a countable subset of* $2^{\mathfrak{c}}$ *and* $I \subset E$*;*
- $F \subset E$ *be a finite subset;*

- *for each* $\xi \in I$, $k_\xi \in \omega$;
- $\{p_\xi : \xi \in I\}$ *be a family of incomparable selective ultrafilters.*
- *for each* $\xi \in I$, $g_\xi : \omega \to ([E]^{<\omega})^{k_\xi}$ *be a function so that* $\{g_\xi^j(m) : j < k_\xi, m \in \omega\}$ *is linearly independent;*
- *for each* $\xi \in I$, $d_\xi \in ([E]^{<\omega})^{k_\xi}$.

Then there exist an increasing sequence $\{b_i : i \in \omega\} \subset \omega$, *a surjective function* $r : \omega \to I$ *and a sequence* $\{E_i : i \in \omega\}$ *of finite subsets of* E *such that:*

a) $F \subset E_0$;
b) $E = \bigcup_{i\in\omega} E_i$;
c) $r(m) \in E_m$ *for each* $m \in \omega$;
d) $\bigcup\{d^j_{r(m)} : j < k_{r(m)}\} \subset E_m$, *for each* $m \in \omega$;
e) $E_{m+1} \supset \bigcup(\{g^j_\xi(b_m) : \xi \in E_m \cap I, j < k_\xi\}) \cup E_m$, *for each* $m \in \omega$;
f) $\{g^j_{r(m)}(b_m) : j < k_{r(m)}\} \cup \{\{\mu\} : \mu \in E_m\}$ *is linearly independent, for each* $m \in \omega$;
g) $\{b_i : i \in r^{-1}(\xi)\} \in p_\xi$, *for every* $\xi \in I$.
Furthermore, if $\{y_n : n \in \omega\} \subset E$ *is faithfully indexed, then* E_i *can be arranged for each* $i \in \omega$ *so that*
h) $\{n \in \omega : y_n \in E_i\} = 2N_i$, *for some* $N_i \in \omega$, *and* $(N_i)_{i\in\omega}$ *is a strictly increasing sequence.*[7]

Proof. Suppose first that I is infinite. Let $E \doteq \{\xi_n : n \in \omega\}$ be an enumeration and $s : \omega \to \omega$ be a strictly increasing function such that $\{\xi_{s(j)} : j \in \omega\} = I$. We will first define a family $\{F_n : n \in \omega\}$ of finite subsets of E. This family will be used to construct the family $\{E_n : n \in \omega\}$.

Choose $N_0 \in \omega$ so that $\left\{n \in \omega : y_n \in F \cup \{\xi_0\} \cup (\bigcup\{d^j_{\xi_{s(0)}} : j < k_{\xi_{s(0)}}\})\right\} \subset 2N_0$, and define

$$F_0 \doteq \{y_n : n \le 2N_0\} \cup F \cup \{\xi_0\} \cup (\bigcup\{d^j_{\xi_{s(0)}} : j < k_{\xi_{s(0)}}\}).$$

Suppose that we have defined finite subsets $F_0, ..., F_l \subset E$ so that

1) $\xi_p \in F_p$ for each $0 \le p \le l$;
2) $F_{p+1} \supset \bigcup(\{g^j_\beta(m) : m \le p,\ \beta \in F_p \cap I,\ j < k_\beta\}) \cup F_p$ for each $0 \le p < l$.
3) $\bigcup\{d^j_{\xi_{s(p)}} : j < k_{\xi_{s(p)}}\} \subset F_p$, for each $0 \le p \le l$.
4) $\{n \in \omega : y_n \in F_p\} = 2N_p$, for some $N_p \in \omega$, for each $0 \le p \le l$.

Now choose $N_{l+1} > N_l$ so that

$$\left\{n \in \omega : y_n \in \bigcup\left(\{g^j_\beta(m) : m \le l,\ \beta \in F_l \cap I,\ j < k_\beta\} \cup \{d^j_{\xi_{s(l+1)}} : j < k_{\xi_{s(l+1)}}\}\right) \cup F_l \cup \{\xi_{l+1}\}\right\} \subset 2N_{l+1},$$

and then define

$$F_{l+1} \doteq \{y_n : n \le 2N_{l+1}\} \cup \bigcup\left(\{g^j_\beta(m) : m \le l,\ \beta \in F_l \cap I,\ j < k_\beta\} \cup \{d^j_{\xi_{s(l+1)}} : j < k_{\xi_{s(l+1)}}\}\right) \cup F_l \cup \{\xi_{l+1}\}.$$

It is clear that 1), 2), 3) and 4) are also satisfied for $F_0, ..., F_{l+1}$. Then, we may construct recursively a family $\{F_n : n \in \omega\}$ of finite subsets of E satisfying 1)-4) for every $p \in \omega$. We also have that $E = \bigcup_{i\in\omega} F_i$.

For each $\xi \in I$ and $n \in \omega$, let

$$A^\xi_n \doteq \{m \in \omega : \{g^j_\xi(m) : j < k_\xi\} \cup \{\{\mu\} : \mu \in F_n\} \text{ is linearly independent}\}.$$

[7] For every $K \in \omega$, $K \ge 2$, we could also arrange E_i for each $i \in \omega$ so that $\{n \in \omega : y_n \in E_i\} \subset KN_i$, for some $N_i \in \omega$, and $(N_i)_{i\in\omega}$ is strictly increasing. The proof would be analogous.

Since $\{g_\xi^j(m) : j < k_\xi, m \in \omega\}$ is linearly independent and F_n is finite, we have that A_n^ξ is cofinite, and then $A_n^\xi \in p_\xi$, for every $n \in \omega$ and $\xi \in I$. Since selective ultrafilters are P-points, for each $\xi \in I$ there exists $A_\xi \in p_\xi$ so that $A_\xi \setminus A_n^\xi$ is finite for every $n \in \omega$.

Now, for each $\xi \in I$, let $v_\xi : \omega \to \omega$ be a strictly increasing function so that $A_\xi \setminus A_n^\xi \subset v_\xi(n)$, for each $n \in \omega$. As every p_ξ is a selective ultrafilter, for each $\xi \in I$ there exists $B_\xi \in p_\xi$ such that

$$B_\xi \cap v_\xi(1) = \emptyset, B_\xi \subset A_\xi \text{ and } |[v_\xi(n)+1, v_\xi(n+1)] \cap B_\xi| \leq 1, \text{ for each } n \in \omega.$$

Let $\{a_n^\xi : n \in \omega\}$ be the strictly increasing enumeration of B_ξ, for each $\xi \in I$. Notice that $a_n^\xi > v_\xi(n) \geq n$ for each $n \in \omega$ and $\xi \in I$. Thus,

$$a_n^\xi \in A_n^\xi, \text{ for each } \xi \in I \text{ and } n \in \omega,$$

and, by Lemma 2.6, there exists a family $\{I_\xi : \xi \in I\}$ of subsets of ω such that:

i) $\{a_i^\xi : i \in I_\xi\} \in p_\xi$ for each $\xi \in I$;
ii) $\{[i, a_i^\xi] : \xi \in I \text{ and } i \in I_\xi\}$ are pairwise disjoint intervals of ω.

By ii), the sets $\{I_\xi : \xi \in I\}$ are pairwise disjoint. We may also assume without loss of generality that $I_{\xi_{s(k)}} \subset \omega \setminus s(k)$ for every $k \in \omega$. Let $\{i_m : m \in \omega\}$ be the strictly increasing enumeration of $\bigcup_{n\in\omega} I_{\xi_{s(n)}}$ and $r : \omega \to I$ be such that $r(m) = \xi_{s(i)}$ if and only if $i_m \in I_{\xi_{s(i)}}$. Define also $b_m \doteq a_{i_m}^{r(m)}$ and $E_m \doteq F_{i_m}$, for each $m \in \omega$.

Conditions a) and b) are trivially satisfied. Moreover, given $m \in \omega$, if $i_m \in I_{\xi_{s(i)}}$, then $i_m \geq s(i)$, and hence $r(m) \in E_m$. Therefore, conditions c) and d) are satisfied. To check condition e), note that $b_m = a_{i_m}^{r(m)} \leq i_{m+1} - 1$ and $E_m = F_{i_m} \subset F_{i_{m+1}-1}$ for each $m \in \omega$, thus

$$\begin{aligned} &E_m \cup \bigcup(\{g_\xi^j(b_m) : \xi \in E_m \cap I, j < k_\xi\}) \\ &\quad \subset F_{i_{m+1}-1} \cup \bigcup(\{g_\xi^j(p) : p \leq i_{m+1}-1,\ \xi \in F_{i_{m+1}-1} \cap I,\ j < k_\xi\}) \\ &\quad \subset F_{i_{m+1}} = E_{m+1}. \end{aligned}$$

Condition f) is also satisfied, since $b_m = a_{i_m}^{r(m)} \in A_{i_m}^{r(m)}$ for each $m \in \omega$, and hence,

$$\{g_{r(m)}^j(b_m) : j < k_{r(m)}\} \cup \{\{\mu\} : \mu \in F_{i_m}\} \text{ is linearly independent.}$$

To check condition g), simply note that, given $\xi \in I$,

$$\{b_m : m \in r^{-1}(\xi)\} = \{a_i^\xi : i \in I_\xi\} \in p_\xi.$$

Condition *h*) follows by construction.

If I is finite, the proof is basically the same, replacing the use of Lemma 2.6 by Lemma 2.5. □

Lemma 2.9. *Let:*

- *Z_0 and Z_1 be disjoint countable subsets of $2^{\mathfrak{c}}$, and $E = Z_0 \cup Z_1$;*
- *$I_0 \subset Z_0$, $I_1 \subset Z_1$, and $I \doteq I_0 \cup I_1$;*
- *$\mathcal{F} \subset [E]^{<\omega}$ be a finite linearly independent subset and, for each $f \in \mathcal{F}$, let $n_f \in 2$;*
- *for each $\xi \in I$, $k_\xi \in \omega$;*

- $\{p_\xi : \xi \in I\}$ *be a family of incomparable selective ultrafilters;*
- *for each* $\xi \in I$, $\delta_\xi = 0$ *if* $\xi \in I_0$ *and* $\delta_\xi = 1$ *if* $\xi \in I_1$;
- *for every* $\xi \in I$, $g_\xi : \omega \to ([Z_{\delta_\xi}]^{<\omega})^{k_\xi}$ *be a function so that* $\{g^j_\xi(m) : j < k_\xi, m \in \omega\}$ *is linearly independent;*
- *for each* $\xi \in I$, $d_\xi \in ([Z_{\delta_\xi}]^{<\omega})^{k_\xi}$;
- $\{z^0_n : n \in \omega\} \subset Z_0$ *and* $\{z^1_n : n \in \omega\} \subset Z_1$ *be sequences of pairwise distinct elements.*

Then, given $(\alpha_0, \alpha_1) \in 2 \times 2$, *there exists a homomorphism* $\Phi : [E]^{<\omega} \to 2$ *such that:*

(i) $\Phi(f) = n_f$, *for every* $f \in \mathcal{F}$;
(ii) for every $\xi \in I$,

$$\Big\{n \in \omega : \Big(\Phi(g^0_\xi(n)), ..., \Phi(g^{k_\xi-1}_\xi(n))\Big) = \Big(\Phi(d^0_\xi), ...\Phi(d^{k_\xi-1}_\xi)\Big)\Big\} \in p_\xi;$$

(iii) $\{n \in \omega : \big(\Phi(\{z^0_n\}), \Phi(\{z^1_n\})\big) = (\alpha_0, \alpha_1)\}$ *is finite.*

Proof. Firstly we apply Lemma 2.8 using the elements given in the hypothesis, $F = \bigcup \mathcal{F}$, and the following sequence $y : \omega \to E$ for item h): for each $n \in \omega$, write $n = 2q + j$ for the unique $q \in \omega$ and $j \in 2$, and put

$$y_{2q+j} = \begin{cases} z^0_q, & \text{if } j = 0 \\ z^1_q, & \text{if } j = 1. \end{cases}$$

Thus we obtain $\{b_i : i \in \omega\} \subset \omega$, $r : \omega \to I$ and $\{E_m : m \in \omega\} \subset [E]^{<\omega}$ satisfying a)–h).

We shall define auxiliary homomorphisms $\Phi_m : [E_m]^{<\omega} \to 2$ inductively. First, we define $\Phi_0 : [E_0]^{<\omega} \to 2$ so that $\Phi_0(f) = n_f$ for each $f \in \mathcal{F}$. Now, suppose that, for $l \in \omega$, we have defined homomorphisms $\Phi_m : [E_m]^{<\omega} \to 2$ for each $m = 0, ..., l$, so that

(1) Φ_{m+1} extends Φ_m for each $0 \le m < l$;
(2) for every $0 \le m < l$,

$$\big(\Phi_{m+1}(g^0_{r(m)}(b_m)), ..., \Phi_{m+1}(g^{k_{r(m)}-1}_{r(m)}(b_m))\big) = \big(\Phi_m(d^0_{r(m)}), ..., \Phi_m(d^{k_{r(m)}-1}_{r(m)})\big);$$

(3) $\big(\Phi_m(\{z^0_n\}), \Phi_m(\{z^1_n\})\big) \neq (\alpha_0, \alpha_1)$ for each $0 < m \le l$ and $n \in \omega$ so that $z^0_n \in E_m \setminus E_{m-1}$.

We shall prove that we may define $\Phi_{l+1} : [E_{l+1}]^{<\omega} \to 2$ so that $\Phi_0, ..., \Phi_{l+1}$ also satisfy (1), (2) and (3). For this, suppose without loss of generality that $r(l) \in I_0$. By item f) of Lemma 2.8, $\{g^j_{r(l)}(b_l) : j < k_{r(l)}\} \cup \{\{\mu\} : \mu \in E_l\}$ is linearly independent, and, by item h), for every $n, m \in \omega$, $z^0_n \in E_m$ if, and only if, $z^1_n \in E_m$. Since $g^j_{r(l)}(b_l) \in [Z_0]^{<\omega}$ for every $j < k_{r(l)}$, and $z^1_n \in Z_1$ for every $n \in \omega$, we conclude that

$$\{\{z^1_n\} : z^0_n \in E_{l+1} \setminus E_l\} \cup \{g^j_{r(l)}(b_l) : j < k_{r(l)}\} \cup \{\{\mu\} : \mu \in E_l\} \qquad (\dagger)$$

is linearly independent. Therefore, using items d) and e) of Lemma 2.8, we may define $\Phi_{l+1} : [E_{l+1}]^{<\omega} \to 2$ extending Φ_l so that

$$(\Phi_{l+1}(g^0_{r(l)}(b_l)), ..., \Phi_{l+1}(g^{k_{r(l)}-1}_{r(l)}(b_l))) = (\Phi_l(d^0_{r(l)}), ..., \Phi_l(d^{k_{r(l)}-1}_{r(l)}))$$

and

$$\Phi_{l+1}(\{z_n^1\}) \neq \alpha_1 \tag{‡}$$

for each $n \in \omega$ such that $z_n^0 \in E_{l+1} \setminus E_l$. Thus, we have that $\Phi_0, ..., \Phi_{l+1}$ also satisfy (1), (2) and (3), and therefore there exists a sequence $(\Phi_m)_{m\in\omega}$ of homomorphisms $\Phi_m : [E_m]^{<\omega} \to 2$ satisfying these properties.

We claim that the homomorphism $\Phi \doteq \bigcup_{n\in\omega} \Phi_n : [E]^{<\omega} \to 2$ satisfies the hypothesis we want. In fact, items (i) and (iii) are clear from the construction and item (ii) follows from the fact that for every $\xi \in I$,

$$(\Phi(g_\xi^0(b_i)), ..., \Phi(g_\xi^{k_\xi-1}(b_i))) = (\Phi(d_\xi^0), ..., \Phi(d_\xi^{k_\xi-1})),$$

for each $i \in r^{-1}(\xi)$, and that $\{b_i : i \in r^{-1}(\xi)\} \in p_\xi$, by item g) of Lemma 2.8. □

The next result is a stronger version of the previous lemma, and uses it in its proof.

Lemma 2.10. *Let:*

- *Z_0 and Z_1 be disjoint countable subsets of $2^{\mathfrak{c}}$, and $E = Z_0 \cup Z_1$;*
- *$I_0 \subset Z_0$, $I_1 \subset Z_1$, and $I \doteq I_0 \cup I_1$;*
- *$\mathcal{F} \subset [E]^{<\omega}$ be a linearly independent finite subset and, for each $f \in \mathcal{F}$, let $n_f \in 2$;*
- *for each $\xi \in I$, $k_\xi \in \omega$;*
- *$\{p_\xi : \xi \in I\}$ be a family of incomparable selective ultrafilters;*
- *for each $\xi \in I$, $\delta_\xi = 0$ if $\xi \in I_0$ and $\delta_\xi = 1$ if $\xi \in I_1$;*
- *for every $\xi \in I$, $g_\xi : \omega \to ([Z_{\delta_\xi}]^{<\omega})^{k_\xi}$ be a function so that $\{g_\xi^j(m) : j < k_\xi,\ m \in \omega\}$ is linearly independent;*
- *for each $\xi \in I$, $d_\xi \in ([Z_{\delta_\xi}]^{<\omega})^{k_\xi}$;*
- *$\{y_n^0 : n \in \omega\} \subset [Z_0]^{<\omega}$ and $\{y_n^1 : n \in \omega\} \subset [Z_1]^{<\omega}$ be linearly independent subsets.*

Suppose that $|Z_i \setminus \bigcup\{y_n^i : n \in \omega\}| = \omega$, for each $i \in 2$. Then, given $(\alpha_0, \alpha_1) \in 2 \times 2$, there exists a homomorphism $\Phi : [E]^{<\omega} \to 2$ such that:

(i) $\Phi(f) = n_f$, *for every* $f \in \mathcal{F}$;
(ii) for every $\xi \in I$,

$$\left\{n \in \omega : \left(\Phi(g_\xi^0(n)), ..., \Phi(g_\xi^{k_\xi-1}(n))\right) = \left(\Phi(d_\xi^0), ..., \Phi(d_\xi^{k_\xi-1})\right)\right\} \in p_\xi;$$

(iii) $\{n \in \omega : (\Phi(y_n^0), \Phi(y_n^1)) = (\alpha_0, \alpha_1)\}$ *is finite.*

Proof. For each $i \in 2$, let $\{z_n^i : n \in \omega\}$ be an enumeration of $\bigcup\{y_n^i : n \in \omega\}$. Next, we extend $\{y_n^i : n \in \omega\}$ to a basis $\mathcal{B}^i$ of $[Z_i]^{<\omega}$ and also $\{\{z_n^i\} : n \in \omega\}$ to a basis $\mathcal{C}^i$ of $[Z_i]^{<\omega}$, for each $i \in 2$. By assumption, $|\mathcal{C}^i \setminus \{\{z_n^i\} : n \in \omega\}| = |\mathcal{B}^i \setminus \{y_n^i : n \in \omega\}| = \omega$, thus we may consider enumerations $\{e_k^i : k \in \omega\}$ of $\mathcal{C}^i \setminus \{\{z_n^i\} : n \in \omega\}$ and $\{f_k^i : k \in \omega\}$ of $\mathcal{B}^i \setminus \{y_n^i : n \in \omega\}$. It is clear that both $\mathcal{B}^0 \cup \mathcal{B}^1$ and $\mathcal{C}^0 \cup \mathcal{C}^1$ are basis of $[E]^{<\omega}$.

Let $\theta : [E]^{<\omega} \to [E]^{<\omega}$ be the isomorphism defined by

$$\theta(y_n^i) = \{z_n^i\},$$

and

$$\theta(f_k^i) = e_k^i,$$

for each $i \in 2$ and $n, k \in \omega$. Note that $\theta|_{[Z_i]^{<\omega}} : [Z_i]^{<\omega} \to [Z_i]^{<\omega}$ is also an isomorphism, for each $i \in 2$.

Let, for every $\xi \in I$, $h_\xi : \omega \to ([Z_{\delta_\xi}]^{<\omega})^{k_\xi}$ be given by $h^j_\xi(n) = \theta(g^j_\xi(n))$ for each $n \in \omega$ and $j < k_\xi$, and $\overline{d}_\xi \in ([Z_{\delta_\xi}]^{<\omega})^{k_\xi}$ be given by $\overline{d}^j_\xi = \theta(d^j_\xi)$, for each $j < k_\xi$.

By Lemma 2.9, there exists a homomorphism $\overline{\Phi} : [E]^{<\omega} \to 2$ so that:

(i) $\overline{\Phi}(\theta(f)) = n_f$, for every $f \in \mathcal{F}$;
(ii) For every $\xi \in I$, $\left\{n \in \omega : \left(\overline{\Phi}(h^0_\xi(n)), ..., \overline{\Phi}(h^{k_\xi-1}_\xi(n))\right) = \left(\overline{\Phi}(\overline{d}^0_\xi), ...\overline{\Phi}(\overline{d}^{k_\xi-1}_\xi)\right)\right\} \in p_\xi$;
(iii) $\{n \in \omega : (\overline{\Phi}(\{z^0_n\}), \overline{\Phi}(\{z^1_n\})) = (\alpha_0, \alpha_1)\}$ is finite.

Thus, the homomorphism $\Phi \doteq \overline{\Phi} \circ \theta : [E]^{<\omega} \to 2$ satisfies the hypothesis we want. □

Remark 1. Note that in the statement of the previous lemma, item (iii) can be replaced by the following (stronger) condition, for a given $\alpha \in 2$:

$$\textit{(iii)}\ \{n \in \omega : \Phi(y^0_n \triangle y^1_n) = \alpha\} \text{ is finite.}$$

Indeed, we could replace condition (†) in the proof of Lemma 2.9 by the fact that

$$\{\{z^0_n, z^1_n\} : z^0_n \in E_{l+1} \setminus E_l\} \cup \{g^j_{r(l)}(b_l) : j < k_{r(l)}\} \cup \{\{\mu\} : \mu \in E_l\}$$

is linearly independent, thus in equation (‡) we could choose

$$\Phi_{l+1}(\{z^0_n, z^1_n\}) \neq \alpha$$

for each $n \in \omega$ such that $z^0_n \in E_{l+1} \setminus E_l$. Then, the proof of Lemma 2.10 would remain the same, just replacing the old condition with the new one when required.

The next result is an easy corollary of the previous lemma.

Corollary 2.11. *Let:*

- *E be a countable subset of $2^{\mathfrak{c}}$;*
- *$I \subset E$;*
- *$\mathcal{F} \subset [E]^{<\omega}$ be a linearly independent finite subset and, for each $f \in \mathcal{F}$, let $n_f \in 2$;*
- *for each $\xi \in I$, $k_\xi \in \omega$.*
- *$\{p_\xi : \xi \in I\}$ be a family of incomparable selective ultrafilters;*
- *for every $\xi \in I$, $g_\xi : \omega \to ([E]^{<\omega})^{k_\xi}$ be a function so that $\{g^j_\xi(m) : j < k_\xi,\ m \in \omega\}$ is linearly independent;*
- *for every $\xi \in I$, $d_\xi \in ([E]^{<\omega})^{k_\xi}$.*

Then there exists a homomorphism $\Phi : [E]^{<\omega} \to 2$ such that:

(i) $\Phi(f) = n_f$, *for every* $f \in \mathcal{F}$;
(ii) For every $\xi \in I$, $\left\{n \in \omega : \left(\Phi(g^0_\xi(n)), ..., \Phi(g^{k_\xi-1}_\xi(n))\right) = \left(\Phi(d^0_\xi), ...\Phi(d^{k_\xi-1}_\xi)\right)\right\} \in p_\xi$.

Although the proof of the following result is similar to the proof of Lemma 2.9 and Lemma 2.10, we present it here for the sake of completeness.

Lemma 2.12. *Let:*

- E *be a countable subset of* $2^{\mathfrak{c}}$*;*
- $I \subset E$*;*
- $\mathcal{F} \subset [E]^{<\omega}$ *be a linearly independent finite subset and, for each* $f \in \mathcal{F}$*, let* $n_f \in 2$*;*
- $n \in \omega$*;*
- $\{p_\xi : \xi \in I\}$ *be a family of incomparable selective ultrafilters;*
- *for every* $\xi \in I$*,* $g_\xi : \omega \to ([E]^{<\omega})^n$ *be a function so that* $\{g_\xi^j(m) : j < n,\ m \in \omega\}$ *is linearly independent;*
- *for every* $\xi \in I$*,* $d_\xi \in ([E]^{<\omega})^n$*;*
- $\{y_k^j : k \in \omega,\ j \leq n\} \subset [E]^{<\omega}$ *be a linearly independent subset.*

Suppose that $|E \setminus \bigcup\{y_k^j : k \in \omega,\ j \leq n\}| = \omega$*. Then, given* $(\alpha_0, ..., \alpha_n) \in 2^{n+1}$*, there exists a homomorphism* $\Phi : [E]^{<\omega} \to 2$ *such that:*

(i) $\Phi(f) = n_f$*, for every* $f \in \mathcal{F}$*;*
(ii) for every $\xi \in I$*,* $\left\{k \in \omega : \left(\Phi(g_\xi^0(k)), ..., \Phi(g_\xi^{n-1}(k))\right) = \left(\Phi(d_\xi^0), ...\Phi(d_\xi^{n-1})\right)\right\} \in p_\xi$*;*
(iii) $\{k \in \omega : (\Phi(y_k^0), ..., \Phi(y_k^n)) = (\alpha_0, ..., \alpha_n)\}$ *is finite.*

Proof. We split the proof in two cases.

Case 1: Suppose that each y_k^j is a singleton, that is, $y_k^j = \{z_k^j\}$, for some $z_k^j \in E$, for every $j \leq n$ and $k \in \omega$.

In this case, we apply Lemma 2.8 using the elements of the statement, $F \doteq \bigcup \mathcal{F}$, $k_\xi = n$ for each $\xi \in I$, and the following sequence $w : \omega \to E$ in item h): for each $m \in \omega$, write $m = (n + 1)q + j$ for the unique $q \in \omega$ and $j \in (n+1)$, and put $w_m = z_q^j$. Thus, we obtain $\{b_i : i \in \omega\}$, $r : \omega \to I$ and $\{E_m : m \in \omega\} \subset [E]^{<\omega}$ satisfying a)-h) of this lemma.

We shall again define auxiliary homomorphisms $\Phi_m : [E_m]^{<\omega} \to 2$, for each $m \in \omega$, inductively. First, define $\Phi_0 : [E_0]^{<\omega} \to 2$ so that $\Phi_0(f) = n_f$, for each $f \in \mathcal{F}$. Suppose that, for $l \in \omega$, we have defined $\Phi_m : [E_m]^{<\omega} \to 2$, for each $m = 0, ..., l$, satisfying that:

(1) Φ_{m+1} extends Φ_m, for each $0 \leq m < l$;
(2) for every $0 \leq m < l$,

$$\left(\Phi_{m+1}(g_{r(m)}^0(b_m)), ..., \Phi_{m+1}(g_{r(m)}^{n-1}(b_m))\right) = \left(\Phi_m(d_{r(m)}^0), ..., \Phi_m(d_{r(m)}^{n-1})\right);$$

(3) $\left(\Phi_m(\{z_k^0\}), ...\Phi_m(\{z_k^n\})\right) \neq \left(\alpha_0, ..., \alpha_n\right)$ for each $0 < m \leq l$ and $k \in \omega$ so that $z_k^0 \in E_m \setminus E_{m-1}$.[8]

Now, since by construction $\{g_{r(l)}^j(b_l) : j < n\} \cup \{\{\mu\} : \mu \in E_l\}$ is linearly independent, we may apply Lemma 2.1 with $A \doteq \{g_{r(l)}^j(b_l) : j < n\}$, $B \doteq \{\{z_k^j\} : z_k^0 \in E_{l+1}, \setminus E_l,\ j \leq n\}$ and $C \doteq \{\{\mu\} : \mu \in E_l\}$ to obtain a subset $B' \subset B$ such that $|B'| \leq |A| = n$ and

$$\{g_{r(l)}^j(b_l) : j < n\} \cup \{\{\mu\} : \mu \in E_l\} \cup (\{\{z_k^j\} : z_k^0 \in E_{l+1} \setminus E_l,\ j \leq n\} \setminus B')$$

is linearly independent. Then, for each $k \in \omega$ so that $z_k^0 \in E_{l+1} \setminus E_l$, there exists $0 \leq j^k \leq n$ such that $z_k^{j^k} \in (\{\{z_k^j\} : z_k^0 \in E_{l+1} \setminus E_l,\ j \leq n\} \setminus B')$. Thus, we may define $\Phi_{l+1} : [E_{l+1}]^{<\omega} \to 2$ extending Φ_l so that

[8] Recall that, by construction, given $k, m \in \omega$, $z_k^j \in E_m$ for **some** $0 \leq j \leq n$ if, and only if, $z_k^j \in E_m$ for **every** $0 \leq j \leq n$.

$$\left(\Phi_{l+1}(g^0_{r(l)}(b_l)), ..., \Phi_{l+1}(g^{n-1}_{r(l)}(b_l))\right) = \left(\Phi_l(d^0_{r(l)}), ..., \Phi_l(d^{n-1}_{r(l)})\right)$$

and

$$\Phi_{l+1}(\{z^{j^k}_k\}) \neq \alpha_{j^k},$$

for every $k \in \omega$ so that $z^0_k \in E_{l+1} \setminus E_l$. Similarly to the proof of Lemma 2.9, we have that $\Phi_0, ..., \Phi_{l+1}$ also satisfy (1)-(3), and therefore there exists a sequence $(\Phi_m)_{m\in\omega}$ of homomorphisms $\Phi_m : [E_m]^{<\omega} \to 2$ satisfying such properties. Again, the homomorphism defined by $\Phi \doteq \bigcup_{n\in\omega} \Phi_n : [E]^{<\omega} \to 2$ satisfies the hypothesis we want.

Case 2: The general case. There is no restriction on elements y^j_k.

Let $\{z_k : k \in \omega\}$ be an enumeration of $\bigcup\{y^j_k : k \in \omega, j \leq n\}$ and $\{z^0_k : k \in \omega\}, ..., \{z^n_k : k \in \omega\}$ be a partition of $\{z_k : k \in \omega\}$. We extend $\{y^j_k : k \in \omega, j \leq n\}$ to a basis $\mathcal{B}$ of $[E]^{<\omega}$ and also $\{\{z^j_k\} : k \in \omega, j \leq n\}$ to a basis $\mathcal{C}$ of $[E]^{<\omega}$. By assumption, $|\mathcal{B} \setminus \{\{z^j_k\} : k \in \omega, j \leq n\}| = |\mathcal{C} \setminus \{y^j_k : k \in \omega, j \leq n\}| = \omega$, thus consider enumerations $\{e_l : l \in \omega\}$ of $\mathcal{B} \setminus \{\{z^j_k\} : k \in \omega, j \leq n\}$ and $\{f_l : l \in \omega\}$ of $\mathcal{C} \setminus \{y^j_k : k \in \omega, j \leq n\}$.

Let $\theta : [E]^{<\omega} \to [E]^{<\omega}$ be the isomorphism defined by:

$$\theta(y^j_k) = \{z^j_k\},$$

for every $k \in \omega$ and $j \leq n$, and

$$\theta(f_l) = e_l,$$

for every $l \in \omega$.

Let also, for each $\xi \in I$, $h_\xi : \omega \to ([E]^{<\omega})^n$ given by $h^i_\xi(m) = \theta(g^i_\xi(m))$, for every $m \in \omega$ and $i < n$, and $\overline{d_\xi} \in ([E]^{<\omega})^n$ given by $\overline{d_\xi}^i = \theta(d^i_\xi)$, for every $i < n$. By the previous case, there exists a homomorphism $\tilde{\Phi} : [E]^{<\omega} \to 2$ so that:

(i) $\tilde{\Phi}(\theta(f)) = n_f$, for each $f \in \mathcal{F}$;
(ii) For every $\xi \in I$, $\left\{m \in \omega : \left(\overline{\Phi}(h^0_\xi(m)), ..., \overline{\Phi}(h^{n-1}_\xi(m))\right) = \left(\overline{\Phi}(\overline{d}^0_\xi), ...\overline{\Phi}(\overline{d}^{n-1}_\xi)\right)\right\} \in p_\xi$;
(iii) $\{k \in \omega : (\overline{\Phi}(\{z^0_k\}), ..., \overline{\Phi}(\{z^n_k\})) = (\alpha_0, ..., \alpha_n)\}$ is finite.

Thus, the homomorphism $\Phi \doteq \overline{\Phi} \circ \theta : [E]^{<\omega} \to 2$ satisfies the hypothesis we want. □

3. A consistent solution to the case $\alpha = \omega$ of the Comfort-like question for countably pracompact groups

Theorem 3.1. *Suppose that there are $\mathfrak{c}$ incomparable selective ultrafilters. Then there exists a (Hausdorff) group G which has all finite powers countably pracompact and such that G^ω is not countably pracompact.*

Proof. The required group will be constructed giving a suitable topology to the Boolean group $[\mathfrak{c}]^{<\omega}$, as follows.

Let $(X_n)_{n>0}$ be a partition of $\mathfrak{c}$ so that $|X_n| = \mathfrak{c}$ for every $n > 0$. For each $n > 0$, let $(X^j_n)_{j<2}$ be a partition of X_n so that

- $|X^0_n| = |X^1_n| = \mathfrak{c}$;
- X^0_n contains only limit ordinals and their next ω elements;
- the initial ω elements of X_n are in X^1_n.

For every $n > 0$, let also

$$Y_n^0 \doteq \{\xi \in X_n^0 : \xi \text{ is a limit ordinal}\},$$

and define the sets $X_0 \doteq \dot{\bigcup}_{n\in\omega} X_n^0$, $X_1 \doteq \dot{\bigcup}_{n\in\omega} X_n^1$ and $Y_0 \doteq \dot{\bigcup}_{n\in\omega} Y_n^0$. Now, consider a family of functions $\{f_\xi : \xi \in Y_0\}$ so that:

1) for each $n > 0$, $\{f_\xi : \xi \in Y_n^0\}$ is an enumeration of all the sequences $(x_k)_{k\in\omega}$ of elements in $([X_n]^{<\omega})^n$ so that $\{x_k^j : k \in \omega, j < n\}$ is linearly independent;
2) given $n > 0$ and $\xi \in Y_n^0$, f_ξ is a function from ω to $([X_n]^{<\omega})^n$ such that $\bigcup_{j<n} \bigcup_{k\in\omega} f_\xi^j(k) \subset \xi$.

Finally, let $\{p_\xi : \xi \in Y_0\}$ be a family of incomparable selective ultrafilters, which exists by hypothesis.

Countable subsets of $\mathfrak{c}$ which have a suitable property of closure related to this construction will be called *suitably closed*[9]:

Definition 3.2. A set $A \in [\mathfrak{c}]^\omega$ is *suitably closed* if, for each $n > 0$ and $\xi \in Y_n^0$ so that $\{\xi + j : j < n\} \cap A \neq \emptyset$, we have that

$$\{\xi + j : j < n\} \cup \bigcup_{j<n} \bigcup_{k\in\omega} f_\xi^j(k) \subset A.$$

Let $\mathcal{A}$ be the set of all homomorphisms $\sigma : [A]^{<\omega} \to 2$, with $A \in [\mathfrak{c}]^\omega$ suitably closed, satisfying that, for every $n > 0$ and $\xi \in A \cap Y_n^0$,

$$\sigma(\{\xi + j\}) = p_\xi - \lim_{k\in\omega} \sigma(f_\xi^j(k)),$$

for each $j < n$.

Enumerate $\mathcal{A}$ by $\{\sigma_\mu : \omega \leq \mu < \mathfrak{c}\}$ and, without loss of generality, we may assume that $\bigcup \mathrm{dom}(\sigma_\mu) \subset \mu$, for each $\mu \in [\omega, \mathfrak{c})$. In what follows, we will construct suitable homomorphisms $\overline{\sigma_\mu} : [\mathfrak{c}]^{<\omega} \to 2$, for every $\mu \in [\omega, \mathfrak{c})$. Note that it is enough to define $\overline{\sigma_\mu}$ in the subset $\{\{\xi\} : \xi \in \mathfrak{c}\}$, since this is a basis for $[\mathfrak{c}]^{<\omega}$.

Firstly, for each $n > 0$, we enumerate all functions $g : S \to 2$ with $S \in [\mathfrak{c}]^{<\omega}$ by $\{g_\xi : \xi \in X_n^1\}$. Without loss of generality, we may assume that $\mathrm{dom}(g_\xi) \subset \xi$, for every $\xi \in X_n^1$, and that for each $g : S \to 2$ as above, $|\{\xi \in X_n^1 : g_\xi = g\}| = \mathfrak{c}$.

Let $\mu \in [\omega, \mathfrak{c})$. If $\xi < \mathfrak{c}$ is such that $\{\xi\} \in \mathrm{dom}(\sigma_\mu)$, we put $\overline{\sigma}_\mu(\{\xi\}) = \sigma_\mu(\{\xi\})$. Otherwise, we have a few cases to consider:

1) if $\xi \in X_1$ and $\mu \in \mathrm{dom}(g_\xi)$, we put $\overline{\sigma}_\mu(\{\xi\}) = g_\xi(\mu)$;
2) if $\xi \in X_1$ and $\mu \notin \mathrm{dom}(g_\xi)$, we put $\overline{\sigma}_\mu(\{\xi\}) = 0$;
3) for the remaining elements of X_0, $\overline{\sigma}_\mu$ is defined recursively, by putting

$$\begin{cases} \overline{\sigma_\mu}(\{\xi + j\}) = p_\xi - \lim_{k\in\omega} \overline{\sigma_\mu}(f_\xi^j(k)) & \text{if } \xi \in Y_n^0 \text{ and } j < n; \\ \overline{\sigma}_\mu(\{\xi\}) = 0, & \text{if } \xi \notin \{\alpha + j : \alpha \in Y_n^0, j < n\}. \end{cases}$$

The definition above uniquely extends each σ_μ to a homomorphism $\overline{\sigma_\mu} : [\mathfrak{c}]^{<\omega} \to 2$, which satisfies that, for each $n > 0$, $\xi \in Y_n^0$ and $j < n$,

[9] The idea of suitably closed sets already appeared in [14], without using a name. Many subsequent works that used Martin's Axiom for countable posets and selective ultrafilters also used this idea. The name *suitably closed* appeared firstly in [13].

$$\overline{\sigma_\mu}(\{\xi + j\}) = p_\xi - \lim_{k\in\omega} \overline{\sigma_\mu}(f_\xi^j(k)). \tag{*}$$

Let now $\overline{\mathcal{A}} \doteq \{\overline{\sigma_\mu} : \omega \leq \mu < \mathfrak{c}\}$ and τ be the weakest (group) topology on $[\mathfrak{c}]^{<\omega}$ making every homomorphism in $\overline{\mathcal{A}}$ continuous. We call this group G. We claim that G is Hausdorff. Indeed, given $x \in [\mathfrak{c}]^{<\omega} \setminus \{\emptyset\}$, let A be a suitably closed set containing x. We may use Corollary 2.11 with $E = A$, $I = A \cap Y_0$, $\mathcal{F} = \{x\}$ and, for each $n > 0$ and $\xi \in Y_n^0 \cap A$, $d_\xi = (\{\xi\}, ..., \{\xi + n - 1\})$, to fix a homomorphism $\sigma : [A]^{<\omega} \to 2$ so that $\sigma \in \mathcal{A}$ and $\sigma(x) = 1$. By construction, there exists $\mu \in [\omega, \mathfrak{c})$ so that $\sigma_\mu = \sigma$, and hence $\overline{\sigma_\mu}(x) = 1$.

Claim 1. *For every $n > 0$, G^n is countably pracompact.*

Proof of the claim. Fix $n > 0$. We claim that $([X_n]^{<\omega})^n \subset G^n$ is a witness to the countable pracompactness property in G^n. Indeed, if U is a nonempty open subset of G, we may fix a function $g : S \to 2$, with $S \in [\mathfrak{c}]^{<\omega}$, so that

$$U \supset \bigcap_{\mu\in S} \overline{\sigma_\mu}^{-1}(g(\mu)).$$

Then, by construction, we may choose $\xi \in X_n^1 \cap (\mu, \mathfrak{c})$ so that $g_\xi = g$, and thus $\{\xi\} \in U$, which shows that $[X_n]^{<\omega}$ is dense in G, and therefore $([X_n]^{<\omega})^n$ is dense in G^n.

We shall now prove that every infinite sequence $\{x_k : k \in \omega\}$ of elements in $([X_n]^{<\omega})^n$ has an accumulation point in G^n. In fact, by Lemma 2.3, there are:

- elements $d_0, ..., d_{n-1} \in [X_n]^{<\omega}$;
- a subsequence $(x_{k_l})_{l\in\omega}$;
- for some $0 \leq t \leq n$, a sequence $(y_l)_{l\in\omega}$ in $([X_n]^{<\omega})^t$
- for each $0 \leq s < n$, a function $P_s : t \to 2$,

satisfying that

i) $x_{k_l}^s = \Big(\sum_{j=0}^{t-1} P_s(j) y_l^j\Big) \triangle d_s$, for every $l \in \omega$ and $0 \leq s < n$.
ii) $\{y_l^j : l \in \omega, 0 \leq j < t\}$ is linearly independent.

By construction, there exists $\xi \in Y_n^0$ so that $f_\xi^j(l) = y_l^j$, for every $l \in \omega$ and $0 \leq j < t$. Since

$$\overline{\sigma_\mu}(\{\xi + j\}) = p_\xi - \lim_{l\in\omega} \overline{\sigma_\mu}(f_\xi^j(l)),$$

for each $\mu \in [\omega, \mathfrak{c})$ and $0 \leq j < n$, we conclude that, for each $0 \leq s < n$,

$$\Big(\sum_{j=0}^{t-1} P_s(j)\{\xi + j\}\Big) \triangle d_s = p_\xi - \lim_{l\in\omega} x_{k_l}^s,$$

and therefore $\{x_k : k \in \omega\}$ has an accumulation point in G^n.[10] □

Claim 2. *G^ω is not countably pracompact.*

[10] In fact, the accumulation point obtained even belongs to $([X_n]^{<\omega})^n$ itself. This shows that the subgroup $[X_n]^{<\omega}$ has its nth-power countably compact, for each $n > 0$.

Proof of the claim. Let $Y \subset G^\omega$ be a dense subset. Consider the set $\{U^j_k : k \in \omega, j \in \omega\}$ of nonempty open subsets of G given by Lemma 2.7. For each $k \in \omega$, we may choose an element $x_k \in Y \cap \prod_{j \leq k} U^j_k \times G^{\omega \setminus k+1}$, and hence

$$\{x^j_k : j \in \omega, k \geq j\}$$

is linearly independent. In what follows, we will show that there exists a subsequence of $\{x_k : k \in \omega\}$ which does not have an accumulation point in G^ω.

For an element $D \in [\mathfrak{c}]^{<\omega}$, we define

$$\mathrm{SUPP}(D) \doteq \{n > 0 : D \cap X_n \neq \emptyset\}.$$

We will split the proof in two cases.

Case 1: There exists $j \in \omega$ so that $\bigcup_{k \in \omega} \mathrm{SUPP}(x^j_k)$ is infinite.

In this case, we may fix a subsequence $\{x^j_{k_m} : m \in \omega\}$ such that

$$\mathrm{SUPP}(x^j_{k_m}) \setminus \left(\bigcup_{p<m} \mathrm{SUPP}(x^j_{k_p}) \right) \neq \emptyset, \tag{1}$$

for every $m \in \omega$. We may also assume that $k_0 \geq j$, and hence $\{x^j_{k_m} : m \in \omega\}$ is linearly independent.

Now we shall show that, for each $x \in G$, x is not an accumulation point of $\{x^j_{k_m} : m \in \omega\}$. First, note that, given $x \in G$, there exists $N_0 \in \omega$ such that, for every $m \geq N_0$,

$$\mathrm{SUPP}(x^j_{k_m}) \setminus \left(\bigcup_{p<m} \mathrm{SUPP}(x^j_{k_p}) \cup \mathrm{SUPP}(x) \right) \neq \emptyset.$$

In fact, since $\mathrm{SUPP}(x)$ is finite and (1) holds, there cannot be infinitely many elements $x^j_{k_m}$ such that $\mathrm{SUPP}(x^j_{k_m}) \subset \bigcup_{p<m} \mathrm{SUPP}(x^j_{k_p}) \cup \mathrm{SUPP}(x)$.

Let

$$F_0 \doteq \bigcup_{p<N_0} \mathrm{SUPP}(x^j_{k_p}) \cup \mathrm{SUPP}(x)$$

and, for $i > 0$,

$$F_i \doteq \mathrm{SUPP}(x^j_{k_{N_0+i-1}}) \setminus \left(\bigcup_{p<N_0+i-1} \mathrm{SUPP}(x^j_{k_p}) \cup \mathrm{SUPP}(x) \right).$$

Define also, for each $i \in \omega$,

$$D_i \doteq \Big(\bigcup_{m \in \omega} x^j_{k_m} \cup x \Big) \cap \Big(\bigcup_{n \in F_i} X_n \Big),$$

and let A_i be a suitably closed set containing D_i such that $A_i \subset \bigcup_{n \in F_i} X_n$. Since $(F_i)_{i \in \omega}$ is a family of pairwise disjoint sets, we have that $(A_i)_{i \in \omega}$ is also a family of pairwise disjoint sets.

Now we may use Corollary 2.11 with: $E = A_0$; $I = A_0 \cap Y_0$; $\mathcal{F} = \{x\}$; and, for every $n > 0$ and $\xi \in Y_n^0 \cap A_0$, $d_\xi = (\{\xi\}, ..., \{\xi + n - 1\})$, to fix a homomorphism $\theta_0 : [A_0]^{<\omega} \to 2$ such that $\theta_0 \in \mathcal{A}$ and $\theta_0(x) = 0$.[11] For $l > 0$, suppose that we have constructed a set of homomorphisms $\{\theta_i : i < l\} \subset \mathcal{A}$ such that

i) $\theta_0(x) = 0$.
ii) θ_i is a homomorphism defined in $\left[\bigcup_{p\leq i} A_p\right]^{<\omega}$ taking values in 2, for each $i < l$.
iii) θ_i extends θ_{i-1} for each $0 < i < l$.
iv) $\theta_i(x^j_{k_{N_0+p}}) = 1$ for each $0 < i < l$ and $p = 0, ..., i - 1$.

Again by Corollary 2.11, we may define a homomorphism $\psi_l : [A_l]^{<\omega} \to 2$ so that $\psi_l \in \mathcal{A}$ and

$$\psi_l\Big(x^j_{k_{N_0+l-1}} \setminus \bigcup_{p<l} D_p\Big) + \theta_{l-1}\Big(x^j_{k_{N_0+l-1}} \cap \bigcup_{p<l} D_p\Big) = 1.$$

Now, since $A_l \cap \bigcup_{i<l} A_i = \emptyset$, we may also define a homomorphism $\theta_l : \left[\bigcup_{p\leq l} A_p\right]^{<\omega} \to 2$ extending both θ_{l-1} and ψ_l. By construction, we have that $\theta_l(x) = 0$ and $\theta_l(x^j_{k_{N_0+p}}) = 1$ for every $p = 0, ..., l-1$. Also, it follows that $\theta_l \in \mathcal{A}$, since $\psi_l \in \mathcal{A}$ and $\theta_i \in \mathcal{A}$ for every $i < l$. Therefore, there exists a family of homomorphisms $\{\theta_l : l \in \omega\} \subset \mathcal{A}$ satisfying i)-iv) for every $l \in \omega$.

Letting $A \doteq \bigcup_{i\in\omega} A_i$ and $\theta \doteq \bigcup_{i\in\omega} \theta_i$, the homomorphism $\theta : [A]^{<\omega} \to 2$ satisfies that $\theta \in \mathcal{A}$, since $\theta_i \in \mathcal{A}$ for each $i \in \omega$. Also, $\theta(x) = 0$ and $\theta(x^j_{k_{N_0+p}}) = 1$ for every $p \in \omega$. By construction, there exists $\mu \in [\omega, \mathfrak{c})$ so that $\theta = \sigma_\mu$, thus $\overline{\sigma_\mu} : [\mathfrak{c}]^{<\omega} \to 2$ satisfies that $\overline{\sigma_\mu}(x^j_{k_m}) = 1$ for each $m \geq N_0$, and $\overline{\sigma_\mu}(x) = 0$. Hence, the element $x \in G$, which was chosen arbitrarily, is not an accumulation point of $\{x^j_{k_m} : m \in \omega\}$. In particular, $\{x_{k_m} : m \in \omega\}$ does not have an accumulation point in G^ω.

Case 2: For every $j \in \omega$, $M_j \doteq \bigcup_{k\in\omega} \mathrm{SUPP}(x^j_k)$ is finite.

In this case, we claim that for each $j \in \omega$ there exists a subsequence $\{k^j_m : m \in \omega\}$ so that, for every $i \leq j$ and $n \in M_i$, either the family $\{x^i_{k^j_m} \cap X_n : m \in \omega\}$ is linearly independent or constant. Indeed, for $j = 0$ and $n_0 \in M_0$, if there exists an infinite subset of $\{x^0_k \cap X_{n_0} : k \in \omega\}$ which is linearly independent, we may fix a subsequence $\{k^{0,0}_m : m \in \omega\}$ so that $\{x^0_{k^{0,0}_m} \cap X_{n_0} : m \in \omega\}$ is linearly independent; otherwise we may fix a subsequence $\{k^{0,0}_m : m \in \omega\}$ so that $\{x^0_{k^{0,0}_m} \cap X_{n_0} : m \in \omega\}$ is constant. Then, if it exists, we may consider another $n_1 \in M_0$ and repeat the process to obtain a subsequence $\{k^{0,1}_m : m \in \omega\}$ which refines $\{k^{0,0}_m : m \in \omega\}$ and satisfies the desired property for n_0 and n_1. Since M_0 is finite, proceeding inductively we may obtain the required subsequence $\{k^0_m : m \in \omega\}$ in the last step. Then, we repeat the process for the next coordinates, always refining the previous subsequence. Now, fix such subsequences $\{k^j_m : m \in \omega\}$, for each $j \in \omega$. We may also suppose that $k^j_0 \geq j$ for each $j \in \omega$.

For each $j \in \omega$, let

$$\overline{M_j} \doteq \{n \in M_j : \{x^j_{k^j_m} \cap X_n : m \in \omega\} \text{ is linearly independent}\}.$$

Note that $\overline{M_j} \neq \emptyset$ for every $j \in \omega$, since $\{x^j_{k^j_m} \cap X_n : m \in \omega, n \in M_j\}$ generates all the elements in the infinite linearly independent set $\{x^j_{k^j_m} : m \in \omega\}$.

Suppose that there exists $j \in \omega$ so that $|\overline{M_j}| > 1$. Fix then $n_0, n_1 \in \overline{M_j}$ distinct. We shall prove that in this case $\{x^j_{k^j_m} : m \in \omega\}$ does not have an accumulation point in G.

For that, consider:

[11] If $x = \emptyset$, $\mathcal{F}$ is not linearly independent and thus we cannot use Corollary 2.11, but it is clear that we can still find such θ_0.

- $x \in G$ chosen arbitrarily;
- $x^0 \doteq x \cap X_{n_0}$, $x^1 \doteq x \cap X_{n_1}$;
- $Z_0 \subset X_{n_0}$ a suitably closed set containing x^0 and $\bigcup\{x^j_{k^j_m} \cap X_{n_0} : m \in \omega\}$, so that $|Z_0 \setminus \bigcup\{x^j_{k^j_m} \cap X_{n_0} : m \in \omega\}| = \omega$;
- $Z_1 \subset X_{n_1}$ a suitably closed set containing x^1 and $\bigcup\{x^j_{k^j_m} \cap X_{n_1} : m \in \omega\}$, so that $|Z_1 \setminus \bigcup\{x^j_{k^j_m} \cap X_{n_1} : m \in \omega\}| = \omega$;
- $\tilde{E} \doteq Z_0 \cup Z_1$;
- $I_0 \doteq Z_0 \cap Y_0 (= Z_0 \cap Y^0_{n_0})$, $I_1 \doteq Z_1 \cap Y_0 (= Z_1 \cap Y^0_{n_1})$ and $I \doteq I_0 \cup I_1$;
- for $\xi \in I$,

$$d_\xi = \begin{cases} (\{\xi\}, ..., \{\xi + n_0 - 1\}), & \text{if } \xi \in I_0 \\ (\{\xi\}, ..., \{\xi + n_1 - 1\}), & \text{if } \xi \in I_1. \end{cases}$$

By Lemma 2.10 and Remark 1, there exists a homomorphism $\tilde{\Phi} : [\tilde{E}]^{<\omega} \to 2$ such that:

(i) for every $s \in x^0 \cup x^1$, $\tilde{\Phi}(\{s\}) = 0$;
(ii) for every $\xi \in I$,

$$\tilde{\Phi}(\{\xi + j\}) = \begin{cases} p_\xi - \lim_{k\in\omega} \tilde{\Phi}(f^j_\xi(k)), \text{ for every } j < n_0, & \text{if } \xi \in I_0 \\ p_\xi - \lim_{k\in\omega} \tilde{\Phi}(f^j_\xi(k)), \text{ for every } j < n_1, & \text{if } \xi \in I_1; \end{cases}$$

(iii) $\left\{m \in \omega : \tilde{\Phi}\left(x^j_{k^j_m} \cap (X_{n_0} \cup X_{n_1})\right) = 0\right\}$ is finite.

Now, fix a suitably closed set E containing $\tilde{E}$, x and $x^j_{k^j_m}$, for each $m \in \omega$, so that $E \cap X_{n_0} = Z_0$ and $E \cap X_{n_1} = Z_1$. Consider the homomorphism $\Phi : [E]^{<\omega} \to 2$ so that, for each $\xi \in E$,

$$\Phi(\{\xi\}) = \begin{cases} \tilde{\Phi}(\{\xi\}), & \text{if } \xi \in \tilde{E} \\ 0, & \text{if } \xi \notin \tilde{E}. \end{cases}$$

In particular, for every $z \in [E]^{<\omega}$ so that $z \cap (X_{n_0} \cup X_{n_1}) = \emptyset$, we have that $\Phi(z) = 0$, and for every $z \in [\tilde{E}]^{<\omega}$, $\Phi(z) = \tilde{\Phi}(z)$.

It follows by construction that $\Phi \in \mathcal{A}$. Furthermore,

$$\begin{aligned} \Phi(x) &= \Phi\Big(\big(x \cap (X_{n_0} \cup X_{n_1})\big) \triangle \big(x \setminus (X_{n_0} \cup X_{n_1})\big)\Big) \\ &= \Phi\Big((x \cap X_{n_0}) \triangle (x \cap X_{n_1})\Big) + \Phi\Big(x \setminus (X_{n_0} \cup X_{n_1})\Big) = \tilde{\Phi}(x^0) + \tilde{\Phi}(x^1) = 0, \end{aligned}$$

and, for every $m \in \omega$,

$$\begin{aligned} \Phi(x^j_{k^j_m}) &= \Phi\Big(\big(x^j_{k^j_m} \cap (X_{n_0} \cup X_{n_1})\big) \triangle \big(x^j_{k^j_m} \setminus (X_{n_0} \cup X_{n_1})\big)\Big) \\ &= \Phi\Big(x^j_{k^j_m} \cap (X_{n_0} \cup X_{n_1})\Big) + \Phi\Big(x^j_{k^j_m} \setminus (X_{n_0} \cup X_{n_1})\Big) = \tilde{\Phi}\Big(x^j_{k^j_m} \cap (X_{n_0} \cup X_{n_1})\Big). \end{aligned}$$

Thus,

$$\left\{m \in \omega : \Phi(x^j_{k^j_m}) = \Phi(x)\right\}$$

is finite. Since, by construction, there exists $\mu \in [\omega, \mathfrak{c})$ so that $\Phi = \sigma_\mu$, we conclude that x cannot be an accumulation point of $\{x^j_{k^j_m} : m \in \omega\}$. As the element $x \in G$ was chosen arbitrarily, the sequence $\{x^j_{k^j_m} : m \in \omega\}$ does not have an accumulation point in G. In particular, $\{x_{k^j_m} : m \in \omega\}$ does not have an accumulation point in G^ω.

Therefore, henceforth we may suppose that $|\overline{M_j}| = 1$ for every $j \in \omega$. We have two subcases to consider.

Case 2.1: There are $j_0, j_1 \in \omega$ distinct so that $\overline{M_{j_0}} \cap \overline{M_{j_1}} = \emptyset$.

Suppose that $j_1 > j_0$, and let $n_0 \in \overline{M_{j_0}}, n_1 \in \overline{M_{j_1}}$. We shall show that the sequence $\{(x^{j_0}_{k^{j_1}_m}, x^{j_1}_{k^{j_1}_m}) : m \in \omega\}$ does not have an accumulation point in G^2. For this, consider:

- $(x^0, x^1) \in G^2$ chosen arbitrarily;
- $y^0 \doteq x^{j_1}_{k^{j_1}_0} \cap X_{n_0}$ and $y^1 \doteq x^{j_0}_{k^{j_1}_0} \cap X_{n_1}$ [12];
- $Z_0 \subset X_{n_0}$ a suitably closed set containing $(x^0 \cup x^1) \cap X_{n_0}$, y^0 and $\bigcup\{x^{j_0}_{k^{j_1}_m} \cap X_{n_0} : m \in \omega\}$, so that $|Z_0 \setminus \bigcup\{x^{j_0}_{k^{j_1}_m} \cap X_{n_0} : m \in \omega\}| = \omega$;
- $Z_1 \subset X_{n_1}$ a suitably closed set containing $(x^0 \cup x^1) \cap X_{n_1}$, y^1 and $\bigcup\{x^{j_1}_{k^{j_1}_m} \cap X_{n_1} : m \in \omega\}$, so that $|Z_1 \setminus \bigcup\{x^{j_1}_{k^{j_1}_m} \cap X_{n_1} : m \in \omega\}| = \omega$;
- $\tilde{E} \doteq Z_0 \cup Z_1$;
- $I_0 \doteq Z_0 \cap Y_0 (= Z_0 \cap Y^0_{n_0})$, $I_1 \doteq Z_1 \cap Y_0 (= Z_1 \cap Y^0_{n_1})$ and $I \doteq I_0 \cup I_1$;
- for $\xi \in I$,

$$d_\xi = \begin{cases} (\{\xi\}, ..., \{\xi + n_0 - 1\}), & \text{if } \xi \in I_0 \\ (\{\xi\}, ..., \{\xi + n_1 - 1\}), & \text{if } \xi \in I_1. \end{cases}$$

By Lemma 2.10, there exists a homomorphism $\tilde{\Phi} : [\tilde{E}]^{<\omega} \to 2$ such that:

(i) for every $s \in (x^0 \cup x^1 \cup y^0 \cup y^1) \cap (X_{n_0} \cup X_{n_1})$, $\tilde{\Phi}(\{s\}) = 0$;
(ii) for every $\xi \in I$,

$$\tilde{\Phi}(\{\xi + j\}) = \begin{cases} p_\xi - \lim_{k \in \omega} \tilde{\Phi}(f^j_\xi(k)), \text{ for every } j < n_0, & \text{if } \xi \in I_0 \\ p_\xi - \lim_{k \in \omega} \tilde{\Phi}(f^j_\xi(k)), \text{ for every } j < n_1, & \text{if } \xi \in I_1; \end{cases}$$

(iii) $\Big\{m \in \omega : \Big(\tilde{\Phi}(x^{j_0}_{k^{j_1}_m} \cap X_{n_0}), \tilde{\Phi}(x^{j_1}_{k^{j_1}_m} \cap X_{n_1})\Big) = (0, 0)\Big\}$ is finite.

Again, fix a suitably closed set E containing $\tilde{E}$, $x^0 \cup x^1$ and $x^{j_0}_{k^{j_1}_m} \cup x^{j_1}_{k^{j_1}_m}$, for each $m \in \omega$, so that $E \cap X_{n_0} = Z_0$ and $E \cap X_{n_1} = Z_1$. Consider the homomorphism $\Phi : [E]^{<\omega} \to 2$ such that, for each $\xi \in E$,

$$\Phi(\{\xi\}) = \begin{cases} \tilde{\Phi}(\{\xi\}), & \text{if } \xi \in \tilde{E} \\ 0, & \text{if } \xi \notin \tilde{E}. \end{cases}$$

It follows by construction that $\Phi \in \mathcal{A}$ and that, for each $i < 2$,

$$\begin{aligned} \Phi(x^i) &= \Phi\Big(\big(x^i \cap (X_{n_0} \cup X_{n_1})\big) \triangle \big(x^i \setminus (X_{n_0} \cup X_{n_1})\big)\Big) \\ &= \Phi\Big((x^i \cap X_{n_0}) \triangle (x^i \cap X_{n_1})\Big) + \Phi\Big(x^i \setminus (X_{n_0} \cup X_{n_1})\Big) = \tilde{\Phi}(x^i \cap X_{n_0}) + \tilde{\Phi}(x^i \cap X_{n_1}) = 0. \end{aligned}$$

[12] Recall that, by construction, the families $\{x^{j_1}_{k^{j_1}_m} \cap X_{n_0} : m \in \omega\}$ and $\{x^{j_0}_{k^{j_1}_m} \cap X_{n_1} : m \in \omega\}$ are constant.

Furthermore, for every $m \in \omega$ and $i < 2$,

$$\begin{aligned}\Phi(x^{j_i}_{k^{j_1}_m}) &= \Phi\Big(\big(x^{j_i}_{k^{j_1}_m} \cap (X_{n_0} \cup X_{n_1})\big) \triangle \big(x^{j_i}_{k^{j_1}_m} \setminus (X_{n_0} \cup X_{n_1})\big)\Big) \\ &= \Phi\Big(x^{j_i}_{k^{j_1}_m} \cap (X_{n_0} \cup X_{n_1})\Big) + \Phi\Big(x^{j_i}_{k^{j_1}_m} \setminus (X_{n_0} \cup X_{n_1})\Big) \\ &= \tilde{\Phi}\Big(x^{j_i}_{k^{j_1}_m} \cap X_{n_0}\Big) + \tilde{\Phi}\Big(x^{j_i}_{k^{j_1}_m} \cap X_{n_1}\Big) = \tilde{\Phi}\Big(x^{j_i}_{k^{j_1}_m} \cap X_{n_i}\Big).\end{aligned}$$

Thus,

$$\Big\{m \in \omega : \Big(\Phi(x^{j_0}_{k^{j_1}_m}), \Phi(x^{j_1}_{k^{j_1}_m})\Big) = (\Phi(x^0), \Phi(x^1)\Big)\Big\}$$

is finite, and therefore $\{(x^{j_0}_{k^{j_1}_m}, x^{j_1}_{k^{j_1}_m}) : m \in \omega\}$ does not have an accumulation point in G^2. In particular, $\{x_{k^{j_1}_m} : m \in \omega\}$ does not have an accumulation point in G^ω.

Case 2.2: For every $j_0, j_1 \in \omega$, $\overline{M_{j_0}} \cap \overline{M_{j_1}} \neq \emptyset$.

In this case, there exists $n_0 > 0$ so that $\overline{M_j} = \{n_0\}$ for every $j \in \omega$. To make the notation simpler, from now on we call $\{k_m : m \in \omega\}$ the sequence $\{k^{n_0}_m : m \in \omega\}$. By construction, $\{x^i_{k_m} : m \in \omega, i \leq n_0\}$ is linearly independent and, for each $i \leq n_0$, there exists $c_i \in [\mathfrak{c}]^{<\omega}$ so that $c_i \cap X_{n_0} = \emptyset$ and $x^i_{k_m} = (x^i_{k_m} \cap X_{n_0}) \triangle c_i$, for every $m \in \omega$. Thus, there exists $m_0 \in \omega$ such that

$$\{x^i_{k_m} \cap X_{n_0} : m \geq m_0,\ i \leq n_0\}$$

is linearly independent.

We shall prove that $\{(x^0_{k_m}, ..., x^{n_0}_{k_m}) : m \in \omega\}$ does not have an accumulation point in G^{n_0+1}. For that, consider:

- $x = (x^0, ..., x^{n_0}) \in G^{n_0+1}$ chosen arbitrarily;
- for each $i = 0, ..., n_0$, $y^i_m = x^i_{k_m} \cap X_{n_0}$ for every $m \geq m_0$.
- $\tilde{E} \subset X_{n_0}$ a suitably closed set containing $(x^0 \cup ... \cup x^{n_0}) \cap X_{n_0}$ and y^i_m, for every $i \leq n_0$ and $m \geq m_0$, so that $|\tilde{E} \setminus \bigcup\{y^i_m : m \geq m_0,\ i \leq n_0\}| = \omega$;
- $I = \tilde{E} \cap Y_0$ $(=\tilde{E} \cap Y^0_{n_0})$;
- for each $\xi \in I$, $d_\xi = (\{\xi\}, ..., \{\xi + n_0 - 1\})$.

By Lemma 2.12, there exists a homomorphism $\tilde{\Phi} : [\tilde{E}]^{<\omega} \to 2$ such that

(i) For every $s \in (x^0 \cup ... \cup x^{n_0}) \cap X_{n_0}$, $\tilde{\Phi}(\{s\}) = 0$.

(ii) For every $\xi \in I$ and $j < n_0$,

$$\tilde{\Phi}(\{\xi + j\}) = p_\xi - \lim_{k \in \omega} \tilde{\Phi}(f^j_\xi(k)).$$

(iii) $\{m \geq m_0 : (\tilde{\Phi}(y^0_m)..., \tilde{\Phi}(y^{n_0}_m)) = (0, ..., 0)\}$ is finite. Note that by construction $\tilde{\Phi}(x^i \cap X_{n_0}) = 0$ for every $i = 0, ..., n_0$.

Consider E a suitably closed set containing $\tilde{E}$, $x^0 \cup ... \cup x^{n_0}$ and $x^0_{k_m} \cup ... \cup x^{n_0}_{k_m}$, for each $m \geq m_0$, so that $E \cap X_{n_0} = \tilde{E}$. Hence, we may define a homomorphism $\Phi : [E]^{<\omega} \to 2$ such that, for each $\xi \in E$,

$$\Phi(\{\xi\}) = \begin{cases} \tilde{\Phi}(\{\xi\}), & \text{if } \xi \in \tilde{E} \\ 0, & \text{if } \xi \notin \tilde{E}. \end{cases}$$

In particular, for every $z \in [E]^{<\omega}$ so that $z \cap X_{n_0} = \emptyset$, we have that $\Phi(z) = 0$. Moreover, for every $z \in [\tilde{E}]^{<\omega}$, $\Phi(z) = \tilde{\Phi}(z)$. Similarly to **Case 2.1**, it follows by construction that $\Phi \in \mathcal{A}$ and that

$$\{m \geq m_0 : (\Phi(x^0_{k_m}), ..., \Phi(x^{n_0}_{k_m})) = (\Phi(x^0), ..., \Phi(x^{n_0}))\} \text{ is finite.}$$

Again, we conclude that x cannot be an accumulation point of $\{(x^0_{k_m}, ..., x^{n_0}_{k_m}) : m \in \omega\}$.

Therefore, in any case, we showed that there exists a subsequence of $\{x_k : k \in \omega\}$ which does not have an accumulation point in G^ω, and thus the group is not countably pracompact. □

As a corollary of the proof above, we obtain:

Corollary 3.3. *Suppose that there are $\mathfrak{c}$ incomparable selective ultrafilters. Then, for each $n \in \omega$, $n > 0$, there exists a (Hausdorff) topological group whose nth power is countably compact and the (n+1)th power is not selectively pseudocompact.*

Proof. With the same notation of the previous proof, for each $n > 0$, we choose the topological subgroup $H \doteq [X_n]^{<\omega} \subset G$. As already mentioned in a footnote, H^n is countably compact. Also, using Lemma 2.12 similarly to what was done in **Case 2.2**, one can show that every sequence $(x^0_k, ..., x^n_k)_{k\in\omega}$ in H^{n+1} so that $\{x^j_k : k \in \omega, j \leq n\}$ is linearly independent does not have an accumulation point in H^{n+1}. Then, it is enough to choose a sequence of nonempty open sets $\{(U^0_k \times ... \times U^n_k) : k \in \omega\} \subset H^{n+1}$, with $\{U^j_k : k \in \omega, j \leq n\}$ as in Lemma 2.7, to prove that H^{n+1} is not selectively pseudocompact. □

Recall that in [10] the authors proved the same result using CH.

4. Consistent solutions to the Comfort-like question for countably pracompact groups in the case of infinite successor cardinals

Theorem 4.1. *Suppose that there are $2^{\mathfrak{c}}$ incomparable selective ultrafilters. Let $\kappa \leq 2^{\mathfrak{c}}$ be an infinite cardinal. Then there exists a (Hausdorff) group G such that G^κ is countably pracompact and G^{κ^+} is not countably pracompact.*

Proof. The required group will be constructed giving a suitable topology to the Boolean group $[2^{\mathfrak{c}}]^{<\omega}$.

Let $\{X_\gamma : \gamma < \kappa\}$ be a partition of $2^{\mathfrak{c}}$ so that $|X_\gamma| = 2^{\mathfrak{c}}$ for every $\gamma < \kappa$. For each $\gamma < \kappa$, we enumerate X_γ in strictly increasing order as $\{x^\gamma_\beta : \beta < 2^{\mathfrak{c}}\}$ (in this case, it is clear that, for every $\gamma < \kappa$ and $\beta < 2^{\mathfrak{c}}$, $\beta \leq x^\gamma_\beta$). Let also:

- $\{J_0, J_1\}$ be a partition of $2^{\mathfrak{c}}$ so that $|J_0| = |J_1| = 2^{\mathfrak{c}}$ and that $\omega \subset J_1$;
- $X^0_\gamma \doteq \{x^\gamma_\beta : \beta \in J_0\}$ and $X^1_\gamma \doteq \{x^\gamma_\beta : \beta \in J_1\}$, for each $\gamma < \kappa$;
- $X_0 \doteq \bigcup_{\gamma<\kappa} X^0_\gamma$ and $X_1 \doteq \bigcup_{\gamma<\kappa} X^1_\gamma$;
- $\mathcal{P} \doteq \{p_\xi : \xi \in J_0\}$ be a family of incomparable selective ultrafilters, which exists by hypothesis.

Now enumerate the set of all injective sequences of $2^{\mathfrak{c}}$ as $\{I_\alpha : \alpha \in J_0\}$, assuming that for every $\alpha \in J_0$, $\mathrm{rng}(I_\alpha) \subset \alpha$. Finally, for each $\alpha \in J_0$ and $\gamma < \kappa$, we define the function $f^\gamma_\alpha : \omega \to [X_\gamma]^{<\omega}$ as

$$f^\gamma_\alpha(l) = \{x^\gamma_{I_\alpha(l)}\},$$

for every $l \in \omega$. Note that, for each $\alpha \in J_0$ and $\gamma < \kappa$, $\mathrm{rng}(f^\gamma_\alpha) \subset [x^\gamma_\alpha]^{<\omega}$.

Next we define which are the suitably closed sets of this construction.

Definition 4.2. A set $A \in [2^{\mathfrak{c}}]^{\omega}$ is suitably closed if, for every $\gamma < \kappa$ and $\beta \in J_0$, if $x^{\gamma}_{\beta} \in A$, then $\bigcup_{l\in\omega} f^{\gamma}_{\beta}(l) \subset A$.

Let $\mathcal{A} = \{\sigma_\mu : \mu \in [\omega, 2^{\mathfrak{c}})\}$ be an enumeration of all homomorphisms $\sigma : [A]^{<\omega} \to 2$, with A suitably closed, such that

$$\sigma(\{x^{\gamma}_{\beta}\}) = p_\beta - \lim_{l\in\omega} \sigma(f^{\gamma}_{\beta}(l)),$$

for every $\gamma < \kappa$ and $\beta \in J_0$ satisfying that $x^{\gamma}_{\beta} \in A$. We may assume without loss of generality that $\bigcup \mathrm{dom}(\sigma_\mu) \subset \mu$, for every $\mu \in [\omega, 2^{\mathfrak{c}})$.

Next, we will properly extend each homomorphism σ_μ in $\mathcal{A}$ to a homomorphism $\overline{\sigma_\mu}$ defined in $[2^{\mathfrak{c}}]^{<\omega}$. For this purpose, we enumerate the set

$$\mathcal{B} \doteq \{\{(\gamma_0, g_0), ..., (\gamma_k, g_k)\} : k \in \omega, |\{\gamma_0, ..., \gamma_k\}| = k + 1, \text{ and, } \\ \text{for each } i = 0, ..., k, \gamma_i < \kappa, \text{ and } g_i : S_i \to 2, \text{ for some } S_i \in [2^{\mathfrak{c}}]^{<\omega}\}$$

as $\{b_\beta : \beta \in J_1\}$. We may also assume that for every $\beta \in J_1$, $b_\beta = \{(\gamma_0, g_0), ..., (\gamma_k, g_k)\}$ is such that $\bigcup_{i=0}^{k} \mathrm{dom}(g_i) \subset \beta$.

Given $\mu \in [\omega, 2^{\mathfrak{c}})$, if $\xi < 2^{\mathfrak{c}}$ is such that $\{\xi\} \in \mathrm{dom}(\sigma_\mu)$, we put $\overline{\sigma_\mu}(\{\xi\}) = \sigma_\mu(\{\xi\})$. Otherwise, we have a few cases to consider. Firstly, we define the homomorphism in the remaining elements of X^1_γ, for each $\gamma < \kappa$, as described in the next paragraph.

Let $\gamma < \kappa$ and $\xi \in X^1_\gamma$ be so that $\{\xi\} \notin \mathrm{dom}(\sigma_\mu)$. Let also $\beta \in J_1$ be the element such that $\xi = x^{\gamma}_{\beta}$. Now,

- if there exists a function $g : S \to 2$, $S \in [2^{\mathfrak{c}}]^{<\omega}$, so that $(\gamma, g) \in b_\beta$ and $\mu \in \mathrm{dom}(g)$, we put $\overline{\sigma_\mu}(\{\xi\}) = g(\mu)$;
- otherwise, we put $\overline{\sigma_\mu}(\{\xi\}) = 0$.

Finally, in the remaining elements of X^0_γ, for each $\gamma < \kappa$, we define $\overline{\sigma_\mu}$ recursively, by putting

$$\overline{\sigma_\mu}(\{x^{\gamma}_{\beta}\}) = p_\beta - \lim_{l\in\omega} \overline{\sigma_\mu}(f^{\gamma}_{\beta}(l)),$$

for each $\beta \in J_0$.

Now we define $\overline{\mathcal{A}} \doteq \{\overline{\sigma_\mu} : \mu \in [\omega, 2^{\mathfrak{c}})\}$. It is clear by the construction that, for each $\mu \in [\omega, 2^{\mathfrak{c}})$,

$$\overline{\sigma_\mu}(\{x^{\gamma}_{\beta}\}) = p_\beta - \lim_{l\in\omega} \overline{\sigma_\mu}(f^{\gamma}_{\beta}(l)),$$

for every $\gamma < \kappa$ and $\beta \in J_0$. Let G be the group $[2^{\mathfrak{c}}]^{<\omega}$ endowed with the topology generated by the homomorphisms in $\overline{\mathcal{A}}$.

Given $x \in G$, we define, similarly as before,

$$\mathrm{SUPP}(x) = \{\gamma < \kappa : x \cap X_\gamma \neq \emptyset\}.$$

We claim that G is Hausdorff. Indeed, let $x \in [2^{\mathfrak{c}}]^{<\omega} \setminus \{\emptyset\}$ and, given $\gamma \in \mathrm{SUPP}(x)$, $z = x \cap X_\gamma$. Let also $A_0 \subset X_\gamma$ be a suitably closed set containing z. In order to use Corollary 2.11, consider:

- $E = A_0$;
- $I = A_0 \cap X^0_\gamma$;
- $\mathcal{F} = \{z\}$;

- $\{q_\xi : \xi \in I\} \subset \mathcal{P}$ so that, for each $\xi \doteq x^\gamma_\beta \in I$, $q_\xi \doteq p_\beta$;

and, for each $\xi \doteq x^\gamma_\beta \in I$,

- $k_\xi = 1$;
- $g_\xi = f^\gamma_\beta$;
- $d_\xi = \{x^\gamma_\beta\}$.

By Corollary 2.11, we may fix a homomorphism $\sigma_0 : [A_0]^{<\omega} \to 2$ so that $\sigma_0 \in \mathcal{A}$ and $\sigma_0(z) = 1$. Now, let A be a suitably closed set containing x so that $A \cap X_\gamma = A_0$ and $\sigma : [A]^{<\omega} \to 2$ be a homomorphism so that

$$\sigma(\{\xi\}) = \begin{cases} \sigma_0(\{\xi\}), & \text{if } \xi \in A_0 \\ 0, & \text{if } \xi \notin A_0. \end{cases}$$

Then, $\sigma \in \mathcal{A}$ and $\sigma(x) = \sigma\big((x \cap X_\gamma) \triangle (x \setminus X_\gamma)\big) = 1$. By construction, there exists $\mu \in [\omega, 2^\mathfrak{c})$ so that $\sigma_\mu = \sigma$, and hence $\overline{\sigma_\mu}(x) = 1$.

Claim 3. *G^κ is countably pracompact.*

Proof of the claim. We claim that $\{(\{x^\gamma_\beta\})_{\gamma<\kappa} : \beta < 2^\mathfrak{c}\} \subset G^\kappa$ is a dense subset for which every infinite subset has an accumulation point in G^κ.

For $k \in \omega$, let $\{\gamma_0, ..., \gamma_k\} \subset \kappa$ be a finite set of size $k+1$ and, for each $i \in \{0, ..., k\}$, let

- $\mu^i_0, ..., \mu^i_{j_i} \in [\omega, 2^\mathfrak{c})$, for some $j_i \in \omega$;
- $g_i : \{\mu^i_0, ..., \mu^i_{j_i}\} \to 2$ be a function.

We shall prove that, if

$$\bigcap_{p=0}^{j_i} (\overline{\sigma_{\mu^i_p}})^{-1}(g_i(\mu^i_p)) \neq \emptyset,$$

for every $i = 0, ..., k$, then there exists $\beta_0 \in J_1$ so that $\{x^{\gamma_i}_{\beta_0}\} \in \bigcap_{p=0}^{j_i} (\overline{\sigma_{\mu^i_p}})^{-1}(g_i(\mu^i_p))$ for each $i = 0, ..., k$. For that, let $\beta_0 \in J_1$ be so that

$$\{(\gamma_0, g_0), ..., (\gamma_k, g_k)\} = b_{\beta_0}.$$

Since, by construction, $\bigcup \mathrm{dom}(\sigma_\mu) \subset \mu$ for every $\mu \in [\omega, 2^\mathfrak{c})$, $\bigcup_{i=0}^k \mathrm{dom}(g_i) \subset \beta_0$ and $\beta_0 \leq x^\gamma_{\beta_0}$ for every $\gamma < \kappa$, it follows that $\overline{\sigma_{\mu^i_p}}(\{x^{\gamma_i}_{\beta_0}\}) = g_i(\mu^i_p)$ for each $i = 0, ..., k$ and $p = 0, ..., j_i$, as we wanted.

Furthermore, given an injective sequence $I_\alpha : \omega \to 2^\mathfrak{c}$, for some $\alpha \in J_0$, we claim that $\{(\{x^\gamma_{I_\alpha(l)}\})_{\gamma<\kappa} : l \in \omega\}$ has $(\{x^\gamma_\alpha\})_{\gamma<\kappa}$ as accumulation point. Indeed, for every $\mu \in [\omega, 2^\mathfrak{c})$, by construction,

$$\overline{\sigma_\mu}(\{x^\gamma_\alpha\}) = p_\alpha - \lim_{l\in\omega} \overline{\sigma_\mu}(\{x^\gamma_{I_\alpha(l)}\}),$$

for each $\gamma < \kappa$. □

Claim 4. *G^{κ^+} is not countably pracompact.*

Proof of the claim. Since the proof of this claim is similar to the proof of **Claim 2** of Theorem 3.1, we omit the details of some arguments.

Let $Z \subset G^{\kappa^+}$ be a dense subset. We shall show that there exists a sequence in Z that does not have an accumulation point in G^{κ^+}. We will again split the proof of this claim in two cases.

Case 1: There exists $j \in \kappa^+$ so that $\bigcup_{z\in Z} \mathrm{SUPP}(z^j)$ is infinite.

In this case, we may fix a sequence $\{z_m : m \in \omega\} \subset Z$ so that

$$\mathrm{SUPP}(z^j_m) \setminus \bigcup_{p<m} \mathrm{SUPP}(z^j_p) \neq \emptyset,$$

for every $m \in \omega$. We shall show that, for a given $y \in G$, y is not an accumulation point of $\{z^j_m : m \in \omega\}$. In particular, this shows that $\{z_m : m \in \omega\}$ does not have an accumulation point in G^{κ^+}.

Remark 2. Although the arguments are analogous to those in **Case 1** of **Claim 2**, as we are about to see, there is another technical complication in this case. We can only guarantee the validity of Corollary 2.11 for suitably closed sets A so that $A \subset X_\gamma$, for some $\gamma < \kappa$. In fact, while the mapping $\xi \in X_0 \cap A \to q_\xi \in \mathcal{P}$ has to be injective,[13] we wish to map x^γ_β, for $\beta \in J_0$ and $\gamma < \kappa$, to p_β.

Let:

1) $M_0 \in \omega$ be such that, for every $m \geq M_0$,

$$\mathrm{SUPP}(z^j_m) \setminus \left(\bigcup_{p<m} \mathrm{SUPP}(z^j_p) \cup \mathrm{SUPP}(y) \right) \neq \emptyset;$$

2) $F_0 \doteq \bigcup_{p<M_0} \mathrm{SUPP}(z^j_p) \cup \mathrm{SUPP}(y)$;
3) for each $i > 0$,

$$F_i \doteq \mathrm{SUPP}(z^j_{M_0+i-1}) \setminus \left(\bigcup_{p<M_0+i-1} \mathrm{SUPP}(z^j_p) \cup \mathrm{SUPP}(y) \right);$$

4) for each $i \in \omega$,

$$D_i \doteq \Big(\bigcup_{m\in\omega} z^j_m \cup y \Big) \cap \Big(\bigcup_{\gamma\in F_i} X_\gamma \Big);$$

5) A_i be a suitably closed set containing D_i such that $A_i \subset \bigcup_{\gamma\in F_i} X_\gamma$;
6) for each $i \in \omega$, $\gamma_i \in F_i$ be arbitrarily chosen;
7) for each $i \in \omega$, $A^0_i \subset X_{\gamma_i}$ be a suitably closed set containing $A_i \cap X_{\gamma_i}$.

In order to use Corollary 2.11, consider

- $E = A^0_0$;
- $I = A^0_0 \cap X^0_{\gamma_0}$;
- $\{q_\xi : \xi \in I\} \subset \mathcal{P}$ so that, for each $\xi \doteq x^{\gamma_0}_\beta \in I$, $q_\xi \doteq p_\beta$;

[13] Indeed, $X_0 \cap A$ will be the set I, in the notation of Corollary 2.11, and then $(q_\xi)_{\xi\in I}$ has to be a family of incomparable selective ultrafilters.

and, for each $\xi \doteq x_\beta^{\gamma_0} \in I$,

- $k_\xi = 1$;
- $g_\xi = f_\beta^{\gamma_0}$;
- $d_\xi = \{x_\beta^{\gamma_0}\}$.

By Corollary 2.11, we may ensure the existence of a homomorphism $\tilde{\theta}_0 : [A_0^0]^{<\omega} \to 2$ such that $\tilde{\theta}_0 \in \mathcal{A}$ and $\tilde{\theta}_0(y \cap X_{\gamma_0}) = 0$. Then, we define $\theta_0 : [A_0]^{<\omega} \to 2$ so that, for every $\xi \in A_0$,

$$\theta_0(\{\xi\}) = \begin{cases} \tilde{\theta}_0(\{\xi\}), & \text{if } \xi \in A_0^0 \\ 0, & \text{if } \xi \notin A_0^0. \end{cases}$$

Note that, in this case, we still have $\theta_0 \in \mathcal{A}$, and also

$$\theta_0(y) = \theta_0(y \cap X_{\gamma_0}) + \theta_0(y \setminus X_{\gamma_0}) = \tilde{\theta}_0(y \cap X_{\gamma_0}) = 0.$$

Suppose that we have constructed a set of homomorphisms $\{\theta_i : i < l\} \subset \mathcal{A}$, for $l > 0$, such that:

i) θ_i is a homomorphism defined in $\left[\bigcup_{p \leq i} A_p\right]^{<\omega}$ taking values in 2, for each $i < l$;
ii) $\theta_0(y) = 0$;
iii) θ_i extends θ_{i-1} for each $0 < i < l$;
iv) $\theta_i(z^j_{M_0+p}) = 1$ for each $0 < i < l$ and $p = 0, ..., i-1$.

Again, in order to use Corollary 2.11, consider:

- $E = A_l^0$;
- $I = A_l^0 \cap X_{\gamma_l}^0$;
- $\{q_\xi : \xi \in I\} \subset \mathcal{P}$ so that, for each $\xi \doteq x_\beta^{\gamma_l} \in I$, $q_\xi \doteq p_\beta$;

and, for each $\xi \doteq x_\beta^{\gamma_l} \in I$,

- $k_\xi = 1$;
- $g_\xi = f_\beta^{\gamma_l}$;
- $d_\xi = \{x_\beta^{\gamma_l}\}$.

By Corollary 2.11, we may ensure the existence of a homomorphism $\tilde{\psi} : [A_l^0]^{<\omega} \to 2$ so that $\tilde{\psi} \in \mathcal{A}$ and

$$\tilde{\psi}\left(z^j_{M_0+l-1} \cap X_{\gamma_l}\right) + \theta_{l-1}\left(z^j_{M_0+l-1} \setminus \bigcup_{\gamma \in F_l} X_\gamma\right) = 1.$$

Then, we define $\psi : [A_l]^{<\omega} \to 2$ so that, for every $\xi \in A_l$,

$$\psi(\{\xi\}) = \begin{cases} \tilde{\psi}(\{\xi\}), & \text{if } \xi \in A_l^0 \\ 0, & \text{if } \xi \notin A_l^0. \end{cases}$$

Let $\theta_l : \left[\bigcup_{p \leq l} A_p\right]^{<\omega} \to 2$ be a homomorphism extending both θ_{l-1} and ψ. By construction, we have that $\theta_l(y) = 0$, $\theta_l \in \mathcal{A}$, and also that

$$\begin{aligned}\theta_l(z^j_{M_0+l-1}) &= \theta_l\Big(z^j_{M_0+l-1} \cap \bigcup_{\gamma\in F_l} X_\gamma\Big) + \theta_l\Big(z^j_{M_0+l-1} \setminus \bigcup_{\gamma\in F_l} X_\gamma\Big)\\ &= \tilde{\psi}\Big(z^j_{M_0+l-1} \cap X_{\gamma_l}\Big) + \psi\Big(z^j_{M_0+l-1} \cap \bigcup_{\gamma\in F_l\setminus\{\gamma_l\}} X_\gamma\Big) + \theta_{l-1}\Big(z^j_{M_0+l-1} \setminus \bigcup_{\gamma\in F_l} X_\gamma\Big)\\ &= \tilde{\psi}\Big(z^j_{M_0+l-1} \cap X_{\gamma_l}\Big) + \theta_{l-1}\Big(z^j_{M_0+l-1} \setminus \bigcup_{\gamma\in F_l} X_\gamma\Big) = 1.\end{aligned}$$

Moreover, it follows by construction that $\theta_l(z^j_{M_0+p}) = \theta_{l-1}(z^j_{M_0+p}) = 1$ for each $0 \le p < l-1$. Therefore, there exists a family of homomorphisms $\{\theta_i : i \in \omega\} \subset \mathcal{A}$ satisfying i)-iv) for every $l \in \omega$.

Letting $A \doteq \bigcup_{i\in\omega} A_i$, the homomorphism $\theta \doteq \bigcup_{i\in\omega} \theta_i : [A]^{<\omega} \to 2$, satisfies that:

- $\theta \in \mathcal{A}$;
- $\theta(y) = 0$;
- $\theta(z^j_{M_0+p}) = 1$ for every $p \in \omega$.

By construction, there exists $\mu \in [\omega, 2^{\mathfrak{c}})$ so that $\theta = \sigma_\mu$, thus $\overline{\sigma_\mu} : [2^{\mathfrak{c}}]^{<\omega} \to 2$ satisfies that $\overline{\sigma_\mu}(z^j_m) = 1$ for each $m \ge M_0$, and $\overline{\sigma_\mu}(y) = 0$. Hence, $y \in G$ is not an accumulation point of $\{z^j_m : m \in \omega\}$.

Case 2: For every $j \in \kappa^+$, $M_j \doteq \bigcup_{z\in Z} \mathrm{SUPP}(z^j)$ is finite.

Since, in this case,

$$\bigcup_{F\in[\kappa]^{<\omega}} \Big\{j \in \kappa^+ : M_j = F\Big\} = \kappa^+,$$

there exists $F_0 \in [\kappa]^{<\omega}$ so that $N \doteq \Big\{j \in \kappa^+ : M_j = F_0\Big\}$ is infinite. Choose $N_0 \subset N$ so that $|N_0| = \omega$, and let $\{j_i : i \in \omega\}$ be an enumeration of N_0.

Now, consider the set $\{U^i_k : k \in \omega, i \in \omega\}$ of nonempty open subsets of G given by Lemma 2.7. For each $k \in \omega$, we may choose an element $z_k \in Z \cap \prod_{i\le k} U^i_k \times G^{\kappa^+\setminus\{j_0,\dots,j_k\}}$. Similarly to what was done in **Case 2** of **Claim 2**, we can fix a subsequence $\{k^i_m : m \in \omega\}$, for each $i \in \omega$, so that:

- $\{k^{i+1}_m : m \in \omega\}$ refines $\{k^i_m : m \in \omega\}$, for each $i \in \omega$;
- for every $i \in \omega$, $p \le i$ and $\gamma \in F_0$, either the family $\{z^{j_p}_{k^i_m} \cap X_\gamma : m \in \omega\}$ is linearly independent or constant;
- $k^i_0 \ge i$, for each $i \in \omega$.

Notice at this point that

$$\Big\{z^{j_i}_{k^i_m} : i \in \omega, m \in \omega\Big\}$$

is linearly independent. For each $i \in \omega$, let

$$\overline{M_{j_i}} \doteq \Big\{\gamma \in F_0 : \{z^{j_i}_{k^i_m} \cap X_\gamma : m \in \omega\} \text{ is linearly independent}\Big\}.$$

Again, we have that $\overline{M_{j_i}} \neq \emptyset$ for every $i \in \omega$. Then, choose $a, b \in \omega$, $b > a$, so that $M \doteq \overline{M_{j_a}} = \overline{M_{j_b}}$. In this case, there exist $c_a, c_b \in [2^{\mathfrak{c}}]^{<\omega}$ so that

$$z^{j_l}_{k^b_m} = \Big(z^{j_l}_{k^b_m} \cap \bigcup_{\gamma\in M} X_\gamma\Big) \triangle c_l,$$

for each $l \in \{a, b\}$ and $m \in \omega$. Thus, there exists $m_0 \in \omega$ so that

$$\left\{ z^{j_l}_{k^b_m} \cap \bigcup_{\gamma \in M} X_\gamma : m \geq m_0,\ l \in \{a, b\} \right\}$$

is linearly independent. By Lemma 2.2, we may fix a subsequence $\{k_m : m \in \omega\}$ of $\{k^b_m : m \in \omega\}$ and $\gamma_0 \in M$ so that

$$\left\{ z^{j_l}_{k_m} \cap X_{\gamma_0} : m \in \omega,\ l \in \{a, b\} \right\}$$

is linearly independent.

We shall show that $\{(z^{j_a}_{k_m}, z^{j_b}_{k_m}) : m \in \omega\}$ does not have an accumulation point in G^2. For this purpose, consider:

- $x = (x^0, x^1) \in G^2$ chosen arbitrarily;
- for each $l \in \{a, b\}$ and $m \in \omega$, $y^l_m \doteq z^{j_l}_{k_m} \cap X_{\gamma_0}$;
- $\tilde{E} \subset X_{\gamma_0}$ a suitably closed set containing $(x^0 \cup x^1) \cap X_{\gamma_0}$ and y^l_m, for each $l \in \{a, b\}$ and $m \in \omega$, so that $|\tilde{E} \setminus \bigcup \{y^l_m : l \in \{a, b\},\ m \in \omega\}| = \omega$;
- $I = \tilde{E} \cap X^0_{\gamma_0}$;
- $\{q_\xi : \xi \in I\} \subset \mathcal{P}$ so that, for each $\xi \doteq x^{\gamma_0}_\beta \in I$, $q_\xi = p_\beta$;
- for each $\xi \doteq x^{\gamma_0}_\beta \in I$, $d_\xi = \{x^{\gamma_0}_\beta\}$;
- for each $\xi \doteq x^{\gamma_0}_\beta \in I$, $g_\xi = f^{\gamma_0}_\beta$.

By Lemma 2.12, there exists a homomorphism $\tilde{\Phi} : [\tilde{E}]^{<\omega} \to 2$ so that:

i) for every $s \in (x^0 \cup x^1) \cap X_{\gamma_0}$, $\tilde{\Phi}(\{s\}) = 0$;
ii) for every $\xi = x^{\gamma_0}_\beta \in I$,

$$\tilde{\Phi}(\{x^{\gamma_0}_\beta\}) = p_\beta - \lim_{l \in \omega} \tilde{\Phi}(f^{\gamma_0}_\beta(l));$$

iii) $\{m \in \omega : (\tilde{\Phi}(y^a_m), \tilde{\Phi}(y^b_m)) = (0, 0)\}$ is finite.

Now, we may consider E a suitably closed set containing $\tilde{E}$, $x^0 \cup x^1$, and $z^{j_l}_{k_m}$, for each $l \in \{a, b\}$ and $m \in \omega$, so that $E \cap X_{\gamma_0} = \tilde{E}$. Let $\Phi : [E]^{<\omega} \to 2$ be the homomorphism such that, for each $\xi \in E$,

$$\Phi(\{\xi\}) = \begin{cases} \tilde{\Phi}(\{\xi\}), & \text{if } \xi \in \tilde{E} \\ 0, & \text{if } \xi \notin \tilde{E}. \end{cases}$$

Then, $\Phi \in \mathcal{A}$,

$$\Phi(x^0) = \Phi(x^1) = 0,$$

and, for each $l \in \{a, b\}$ and $m \in \omega$,

$$\Phi(z^{j_l}_{k_m}) = \tilde{\Phi}(y^l_m) + \Phi(z^{j_l}_{k_m} \setminus X_{\gamma_0}) = \tilde{\Phi}(y^l_m).$$

Thus, we conclude that

$$\left\{ m \in \omega : (\Phi(z^{j_a}_{k_m}), \Phi(z^{j_b}_{k_m})) = (\Phi(x^0), \Phi(x^1)) \right\}$$

is finite. Since, by construction, there exists $\mu \in [\omega, 2^{\mathfrak{c}})$ so that $\sigma_\mu = \Phi$, we conclude that x cannot be an accumulation point of $\{(z^{j_a}_{k_m}, z^{j_b}_{k_m}) : m \in \omega\}$. Since $x \in G^2$ is arbitrary, $\{z_{k_m} : m \in \omega\} \subset Z$ does not have an accumulation point in G^{κ^+}.

Therefore, G^{κ^+} is not countably pracompact. □

5. Some remarks and questions

As we already mentioned in the first section, in the case of selectively pseudocompact groups, the Comfort-like Question 1.10 is still not solved consistently only for the case $\alpha = \omega$:

Question 5.1. Is there a topological group G so that G^k is selectively pseudocompact for every $k \in \omega$, but G^ω is not selectively pseudocompact?

In the case there is a positive consistent answer to the above question, one can also ask:

Question 5.2. Is there a topological group G so that G^k is countably compact for every $k \in \omega$, but G^ω is not selectively pseudocompact?

Regarding countably pracompact topological groups, Theorem 4.1 for $\kappa = 2^{\mathfrak{c}}$ shows that there exists a group G so that $G^{2^{\mathfrak{c}}}$ is countably pracompact but $G^{(2^{\mathfrak{c}})^+}$ is not countably pracompact. Interestingly, for countably compact spaces we know that this is not the case: given a Hausdorff topological space X, if $X^{2^{\mathfrak{c}}}$ is countably compact, then X^α is countably compact for every $\alpha > 2^{\mathfrak{c}}$. Thus, it may be interesting to study the following questions further. The first one is a stronger version of Question 1.11 for $\alpha = \omega$, which we solved in this paper.

Question 5.3. Is there a topological group G so that G^k is countably compact for every $k \in \omega$ and G^ω is not countably pracompact?

Question 5.4. For which limit cardinals $\omega < \alpha \leq 2^{\mathfrak{c}}$ is there a topological group G such that G^γ is countably pracompact for every cardinal $\gamma < \alpha$, but G^α is not countably pracompact?

Question 5.5. For which cardinals $\alpha > (2^{\mathfrak{c}})^+$ is there a topological group G such that G^γ is countably pracompact for all cardinals $\gamma < \alpha$, but G^α is not countably pracompact?

Also, it is natural to ask which the *stopping point* is, if any:

Question 5.6. Is there a cardinal κ such that, for each topological group G, G^κ countably pracompact implies that G^γ is countably pracompact for every $\gamma > \kappa$?

In ZFC, as mentioned, we do not even know answers to the following questions.

Question 5.7 *(ZFC).*

a) Is there a selectively pseudocompact group whose square is not selectively pseudocompact?
b) **(stronger version)** Is there a countably compact group whose square is not selectively pseudocompact?

Question 5.8 *(ZFC).*

a) Is there a countably pracompact group whose square is not countably pracompact?
b) **(stronger version)** Is there a countably compact group whose square is not countably pracompact?

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